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# Phases of $\mathcal{N} = 1$ Supersymmetric $SO/Sp$ Gauge Theories via Matrix Model

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## Abstract

We extend the results of Cachazo, Seiberg and Witten to $\mathcal{N} = 1$ supersymmetric gauge theories with gauge groups $SO(2N), SO(2N + 1)$ and $Sp(2N)$. By taking the superpotential which is an arbitrary polynomial of adjoint matter $\Phi$ as a small perturbation of $\mathcal{N} = 2$ gauge theories, we examine the singular points preserving $\mathcal{N} = 1$ supersymmetry in the moduli space where mutually local monopoles become massless. We derive the matrix model complex curve for the whole range of the degree of perturbed superpotential. Then we determine a generalized Konishi anomaly equation implying the orientifold contribution. We turn to the multiplication map and the confinement index $K$ and describe both Coulomb branch and confining branch. In particular, we construct a multiplication map from $SO(2N+1)$ to $SO(2KN-K+2)$ where $K$ is an even integer as well as a multiplication map from $SO(2N)$ to $SO(2KN-2K+2)$ ($K$ is a positive integer), a map from $SO(2N+1)$ to $SO(2KN-K+2)$ ($K$ is an odd integer) and a map from $Sp(2N)$ to $Sp(2KN+2K-2)$. Finally we analyze some examples which show some duality: the same moduli space has two different semiclassical limits corresponding to distinct gauge groups.

# 1 Introduction

The $\mathcal{N}=1$ supersymmetric gauge theories in four dimensions are of considerable interest and have a rich physical contents. By studying the structure of quantum moduli space, one gets some of very important nonperturbative phenomena such as confinement and dynamical symmetry breaking because many nonperturbative features are related to the vacua such as mass generation and fermion condensates and so on. In particular, in these theories, there exists a special class of observables that are constrained by holomorphy, for example, holomorphic effective superpotential. The holomorphy of the quantum superpotential makes it possible to determine the light degree of freedom and hence the quantum moduli space. Although the exact results for many different models have been obtained, a general recipe for computing the effective superpotential has remained to be understood fully.

Partially wrapping D-branes over nontrivial cycles of the geometries yield large classes of interesting gauge theories, depending on the choice of geometries. Vafa and his collaborators shed some light on this problem in the viewpoint of gravity dual [1, 2, 3, 4]. They considered four dimensional gauge theories on the world volume of D5-brane wrapped on $\mathbf{S}^2$ and claimed, as far as the computation of the effective superpotential is concerned, that this geometry can be replaced by a dual geometry where the $\mathbf{S}^2$ is replaced by the $\mathbf{S}^3$ and D5-branes by RR fluxes through these cycles. As discussed in [5, 6, 7], these RR fluxes generate the effective superpotential that is equivalent to the effective superpotential of four dimensional gauge theories on the worldvolme of D5-brane. The equivalence and validity of this duality have been tested and proved for several models in many papers [2, 8, 9, 10, 11, 12, 13, 14, 15, 16]. In this context, the effective superpotential can be explicitly expanded in terms of a period integral of the geometry that can be interpreted as a glueball superfield in four dimensional gauge theories.

Dijkgraaf and Vafa gave a new recipe to these calculations of the effective superpotentials [17, 18, 19, 20]. They claimed that perturbative analysis of effective superpotential was captured by the matrix model perturbation. Thus what we have to do is to sum up Feynman diagram of the matrix model. If one wants $n$-th order instanton effect, one has only to compute Feynman diagram up to $(n-1)$ loop. In this context, they also claimed that the loop equation of matrix model, that plays an important role in the matrix model, is equivalent to the Riemann surface that comes from the dual geometry discussed above. After all, this Riemann surface leads to a fruitful system for studying the holomorphic information of four dimensional $\mathcal{N}=1$ gauge theories. After their works, the correspondence between the several matrix models and four dimensional gauge theories attracted wide attention, and many papers, which include the extension to other gauge groups and an adding flavors and so on, appeared in [21]-[74]. In particular, in [54, 69], Ferrari has discussed the quantum parameter space of the $\mathcal{N}=1$ $U(N)$ gauge theory with one adjoint matter $\Phi$ and a cubic tree level superpotential. These works are one of the motivations of our paper.

Recently in [25] they showed that these matrix model analysis could be interpreted within purely field theoretic point of view. In particular for the $U(N)$ gauge theory with adjoint matter $\Phi$ and a polynomial superpotential $W(\Phi)$, a generalized Konishi anomaly equation, providing both a connection between the quark condensation and the gluino condensation in the supersymmetric gauge theories and possible ways to explore the nonperturbative aspects of the supersymmetric gauge theories, gives rise to the loop equation of matrix model. In other words, within gauge theories there is some aspect that can be interpreted as matrix models.

Later, Cachazo, Seiberg and Witten [26] have discussed a new kind of duality. By changing the parameters of $W(\Phi)$, one can transit several vacua with different broken gauge groups continuously and holomorphically. There was no restrictions to the degree of the superpotential $W(\Phi)$ which can be an arbitrary. They derived the matrix model curve and showed a generalized Konishi anomaly equation from the strong gauge coupling approach where the superpotential is considered as a small perturbation of a strongly coupled $\mathcal{N}=2$ gauge theory with $W=0$.

In this paper, we extend the analysis given by Cachazo, Seiberg and Witten to $\mathcal{N}=1$ $SO/Sp$ gauge theories with arbitrary tree level superpotentials. We survey the phase structures of the corresponding gauge theories: Which vacua allow us to smooth transition? As in $U(N)$ case, are there Coulomb vacua in $SO/Sp$ gauge theories? We answer these questions in this paper.

In section 2.1, we analyze $SO(2N)$ gauge theories in terms of strong gauge coupling approach. Since we deal with $\mathcal{N}=2$ theories deformed by tree level superpotential, we have to constrain $\mathcal{N}=2$ Coulomb vacua to the special points where monopoles or dyons become massless. By using $\mathcal{N}=2$ curve together with this constraint that will be incorporated to the effective superpotential and by applying the usual contour integral formulas heavily, we will derive matrix model curve which will be used to determine the number of vacua for fixed tree level superpotential and a generalized Konishi anomaly equation where we will see the orientifold contribution, peculiar to the gauge group, which does not vanish even in the semi-classical limit, for arbitrary degree of superpotential. In order to produce these, we are looking for the equations of motion about the Lagrange multipliers, parameters of moduli space and the locations of massless monopoles for given effective superpotential which depends on these fields, the couplings $g_{2r}$ and the scale $\Lambda$. Although the discussions of matrix model curve in different context have already been studied for special case in [8, 11], we extend to the analysis for general case.

In section 2.2, we repeat the analysis given in section 2.1 for $SO(2N+1)$ gauge theories for completeness. We also derive matrix model curve and a generalized Konishi anomaly equation for arbitrary degree of superpotential. Remembering the power behavior of $x$ and $\Lambda$ in $\mathcal{N}=2$ curve carefully, we also see the orientifold contribution, with different strength, which does not vanish even in the semiclassical limit.

In section 2.3, we discuss a confinement index $K$ and multiplication maps for $SO(N)$ gauge theories. These ideas play the central role for the study of phase structure. Since we can construct a multiplication map from $SO(2N+1)$ to $SO(2KN-K+2)$ where $K$ is even integer, this implies that $SO(2N+1)$ gauge theory where the number of colors is odd and $SO(2KN-K+2)$ gauge theory where the number of colors is even are closely related to each other. There are also a multiplication map from $SO(2N)$ to $SO(2KN-2K+2)$ where $K$ is a positive integer and a multiplication map from $SO(2N+1)$ to $SO(2KN-K+2)$ where $K$ is odd integer. In checking these multiplication maps, the properties of first and second kinds of Chebyshev polynomials and the monopole constraints we will describe are crucial. Moreover it does not seem to allow one can construct the multiplication map from $SO(2N)$ to $SO(2M+1)$. Therefore, for $SO(2N)$ and $SO(2N+1)$ gauge theories, there exist only *three* possible ways to have multiplication maps. The criterion of a confining phase and a Coulomb phase given in [26] is extended to $SO(2N)$ and $SO(2N+1)$ gauge theories. Since the special combination $(N-2)$ for $SO(N)$ gauge theory, that is dual Coxeter number, coming from the contribution of unoriented diagram giving rise to $-2$ term in the effective superpotential provides the number of vacua of the gauge theories, we can use the criterion of $U(N)$ case [26] straightforwardly and equivalently.

In section 2.4, we analyze several explicit examples illustrating what we have discussed in section 2.3. We study examples with gauge group $SO(N)$ for $N=4,5,6,7,8$. In particular, $SO(6)$ gauge theory provides four confining vacua by analyzing the matrix model curve by turning on a superpotential. According to the scheme of a multiplication map from $SO(2N)$ to $SO(2KN-2K+2)$ we have discussed in section 2.3 and by transforming the characteristic polynomial with confinement index $K=2$ and $N=3$, we confirm these vacua are really confining phase. Moreover in $SO(8)$ gauge theory, we can see a smooth transition between different classical gauge group and multiplication map from $SO(5)$ by looking at the confining phase, based on a map from $SO(2N+1)$ to $SO(2KN-K+2)$ described in section 2.3 where $N=2$ and $K=2$.

In section 3.1, as in section 2.1, by restricting $\mathcal{N}=2$ theories to the special points where monopoles become massless we discuss $\mathcal{N}=1$ $Sp(2N)$ gauge theories with arbitrary tree level superpotential. we will derive matrix model curve which will be used to determine the number of vacua for fixed tree level superpotential and a generalized Konishi anomaly equation where we will see also the orientifold contribution, peculiar to the $Sp(2N)$ group, which does not vanish even in the semiclassical limit, for arbitrary degree of superpotential. Comparing with the $SO(2N)$ gauge theory we have discussed in section 2.1, there exists exactly same contribution with an extra minus sign. In order to produce these, we are studying for the equations of motion about the Lagrange multipliers, parameters of moduli space and the locations of massless monopoles for given effective superpotential.

In section 3.2, we discuss a confinement index and multiplication maps for $Sp(2N)$ gauge theories similarly. Since we can construct a multiplication map from $Sp(2N)$ to $Sp(2KN+2K-2)$ where $K$ is a positive integer, this implies that $Sp(2N)$ gauge theory and $Sp(2KN+2K-2)$ gauge theory are related to each other. That is the number of vacua of $Sp(2KN+2K-2)$ gauge theory is exactly the confinement index $K$ multiplied by those of $Sp(2N)$ gauge theory Coulomb phase. In doing this, we will use various properties of Chebyshev polynomials and monopole constraints frequently.

In section 3.3, we analyze several explicit examples describing what we have discussed in section 3.2. We study examples with gauge group $Sp(N)$ for $N = 2, 4, 6$. In particular, the $Sp(6)$ gauge theory provides a smooth transition between different classical gauge groups, based on a map from $Sp(2N)$ to $Sp(2KN+2K-2)$ described in section 3.2 where $N = 1$ and $K = 2$.

Although our work is based on [26] completely and there are not too much originality, we hope our work is illuminating and the results we have found newly provide a better understanding of the structures of vacua of the gauge theories.

# 2 $SO(2N)$ and $SO(2N+1)$ gauge theories

## 2.1 Strong gauge coupling approach:$SO(2N)$ case

The strong gauge coupling approach was studied in [8] by using the method of [82, 85, 86]. Now we review it and extend it to the case where some of $N_i$ (see (2.26) for the notation) vanish and to allow the more general superpotential in which the degree of it can be arbitrary without any restrictions. Let us consider the superpotential regarded as a small perturbation of an $\mathcal{N} = 2$ $SO(2N)$ gauge theory [75, 76, 77, 78, 79, 80, 81, 82, 8, 11]

$$W(\Phi) = \sum_{r=1}^{k+1} \frac{g_{2r}}{2r} \mathrm{Tr}\Phi^{2r} \equiv \sum_{r=1}^{k+1} g_{2r} u_{2r}, \qquad u_{2r} \equiv \frac{1}{2r} \mathrm{Tr}\Phi^{2r} \tag{2.1}$$

where $\Phi$ is an adjoint scalar chiral superfield and we denote its eigenvalues by $\pm\phi_I (I = 1, 2, \cdots, N)$ that are purely imaginary values. The degree of the superpotential $W(\Phi)$ is $2(k+1)$. Since $\Phi$ is an antisymmetric matrix we can transform to the following simple form,

$$\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathrm{diag}(i\phi_1, \cdots, i\phi_N). \tag{2.2}$$

When we replace $\mathrm{Tr}\Phi^{2r}$ with $\langle \mathrm{Tr}\Phi^{2r} \rangle$, the superpotential becomes the effective superpotential. We introduce a classical $2N \times 2N$ matrix $\Phi_{cl}$ such that $\langle \mathrm{Tr}\Phi^{2r} \rangle = \mathrm{Tr}\Phi_{cl}^{2r}$ for $r = 1, 2, \cdots, N$. Also $u_{2r} \equiv \frac{1}{2r} \mathrm{Tr}\Phi_{cl}^{2r}$ that are independent. However, for $2r > 2N$, both $\mathrm{Tr}\Phi^{2r}$ and $\langle \mathrm{Tr}\Phi^{2r} \rangle$ can be written as the $u_r$ of $2r \leq 2N$. The classical vacua can be obtained by putting all the eigenvalues of $\Phi$ and $\Phi_{cl}$ equal to the roots of $W'(z) = \sum_{r=1}^{k+1} g_{2r} z^{2r-1}$. We will take the degree of superpotential to be $2(k+1) \leq 2N$ first in which the $u_r$ are independent and $\langle \mathrm{Tr}\Phi^{2r} \rangle = \mathrm{Tr}\Phi_{cl}^{2r}$. Then we will take the degree of superpotential to be arbitrary.

At a generic point on the Coulomb branch of the $\mathcal{N}=2$ theory, the low energy gauge group is $U(1)^N$. We study the vacua in which the perturbation by $W(\Phi)$ (2.1) remains only $U(1)^n$ gauge group at low energies, with $2n \leq 2k$. This happens if the remaining degrees of freedom become massive for nonzero $W$ due to the condensation of $(N-n)$ mutually local magnetic monopoles or dyons. This occurs only at points where at $W'=0$ the monopoles are massless on some particular submanifold $< u_{2r} >$. This can be done by including the $(N-n)$ monopole hypermultiplets in the superpotential. Then the exact effective superpotential by adding (2.1), near a point with $(N-n)$ massless monopoles, is given by

$$W_{eff} = \sqrt{2} \sum_{l=1}^{N-n} M_l(u_{2r}) q_l \widetilde{q}_l + \sum_{r=1}^{k+1} g_{2r} u_{2r}.$$

Here $q_l$ and $\widetilde{q}_l$ are the monopole fields and $M_l(u_{2r})$ is the mass of $l$-th monopole as a function of the $u_{2r}$. The variation of $W_{eff}$ with respect to $q_l$ and $\widetilde{q}_l$ vanishes. However, the variation of it with respect to $u_{2r}$ does not lead to the vanishing of $q_l \widetilde{q}_l$ and the mass of monopoles should vanish for $l = 1, 2, \cdots, (N-n)$ in a supersymmetric vacuum. Therefore the superpotential in this supersymmetric vacuum becomes $W_{exact} = \sum_{r=1}^{k+1} g_{2r} < u_{2r} >$. The masses $M_i$ are equal to the periods of some meromorphic one-form over some cycles of the $\mathcal{N}=2$ hyperelliptic curve.

It is useful and convenient to consider a singular point in the moduli space where $(N-n)$ mutually local monopoles are massless. Then the $\mathcal{N}=2$ curve of genus $(2N-1)$ degenerates to a curve of genus $2n$ [8, 11] (See also [83]) and it is given by

$$y^2 = P_{2N}^2(x) - 4\Lambda^{4N-4} x^4 = x^2 H_{2N-2n-2}^2(x) F_{2(2n+1)}(x), \tag{2.3}$$

where

$$H_{2N-2n-2}(x) = \prod_{i=1}^{N-n-1} (x^2 - p_i^2), \quad F_{2(2n+1)}(x) = \prod_{i=1}^{2n+1} (x^2 - q_i^2)$$

where $H_{2N-2n-2}(x)$ is a polynomial in $x$ of degree $(2N-2n-2)$ that gives $(2N-2n-2)$ double roots and $F_{2(2n+1)}(x)$ is a polynomial in $x$ of degree $(4n+2)$ that is related to the deformed superpotential (2.4). Both functions are even functions in $x$. That is, a function of $x^2$ which is peculiar to the gauge group $SO(2N)$. The characteristic function $P_{2N}(x)$ that is also function of $x^2$ is described in terms of the eigenvalues of $\Phi$ (2.2) as follows:

$$P_{2N}(x) \quad = \quad \det(x - \Phi_{cl}) = \prod_{I=1}^{N} (x^2 - \phi_I^2).$$

The degeneracy of the above curve can be checked by computing both $y^2$ and $\frac{\partial y^2}{\partial x^2}$ at the point $x = \pm p_i$ and $x = 0$ obtaining a zero. The factorization condition (2.3) can be described and encoded by Lagrange multipliers [26]. When the degree $(2k+1)$ of $W'(x)$ is equal to $(2n+1)$, the highest $(2n+2)$ coefficients of $F_{2(2n+1)}(x)$ are given in terms of $W'(x)$ as follows [8, 11]:

$$F_{2(2n+1)}(x) = \frac{1}{g_{2n+2}^2} W'_{2n+1}(x)^2 + \mathcal{O}(x^{2n}), \qquad 2k = 2n. \tag{2.4}$$

The superpotential $W(x)$ is known and one should know the $\mathcal{N}=2$ vacua or $P_{2N}(x)$ through (2.3). The relation (2.4) determines $F_{2(2n+1)}(x)$ in terms of $(2n+2)$ undetermined coefficients coming from $\mathcal{O}(x^{2n})$. Then these can be obtained by demanding the existence of a polynomial $H_{2N-2n-2}(x)$ via (2.3). In this case, there exists a unique solution. We return to the explicit examples illustrating the mechanism of this in section 2.4.

- **Superpotential of degree $2(k+1)$ less than $2N$**

Now we study (2.4) when $2k > 2n$ by introducing the constraints. We follow the basic idea of [26] and repeat the derivations of (2.4) and generalize to the arbitrary degree of superpotential. That is, in the range $2n+2 \leq 2k+2 \leq 2N$, let us consider the superpotential under these constraints (2.3),

$$\begin{aligned} W_{eff} &= \sum_{r=1}^{k+1} g_{2r} u_{2r} \\ &+ \sum_{i=0}^{2N-2n-2} \left[ L_i \oint \frac{P_{2N}(x) - 2\epsilon_i x^2 \Lambda^{2N-2}}{(x-p_i)} dx + B_i \oint \frac{P_{2N}(x) - 2\epsilon_i x^2 \Lambda^{2N-2}}{(x-p_i)^2} dx \right] \end{aligned} \tag{2.5}$$

where $L_i$ and $B_i$ are Lagrange multipliers imposing the constraints and $\epsilon_i = \pm 1$. The contour integration encloses all $p_i$'s and the factor $1/2\pi i$ is absorbed in the symbol of $\oint$ for simplicity. The $p_i$'s where $i = 0, 1, 2, \cdots, (2N-2n-2)$ are the locations of the double roots of $y^2 = P_{2N}^2(x) - 4\Lambda^{4N-4}x^4$. The $P_{2N}(x)$ depends on $u_{2r}$. Note that massless monopole points appear in pair $(p_i, -p_i)$ where $i = 1, 2, \cdots, (N-n-1)$. So we denote half of the $p_i$'s by $p_{N-n-1+i} = -p_i, i = 1, 2, \cdots, (N-n-1)$. Moreover we define $p_0 = 0$. Since $P_{2N}(x)$ is an even function, if the following constraints are satisfied at $x = p_i$ where $i = 1, 2, \cdots, (N-n-1)$,

$$\left( P_{2N}(x) - 2x^2 \epsilon_i \Lambda^{2N-2} \right)|_{x=p_i} = 0, \qquad \frac{\partial}{\partial x} \left( P_{2N}(x) - 2x^2 \epsilon_i \Lambda^{2N-2} \right)|_{x=p_i} = 0,$$

then they also automatically are satisfied at $x = -p_i$. Then the numbers of constraint that we should consider are $(N-n-1)$. Thus we denote the half of the Lagrange multipliers by $L_{N-n-1+i} = L_i$ and $B_{N-n-1+i} = B_i$ where $i = 1, 2, \cdots, (N-n-1)$. Due to the fact that the second derivative of $P_{2N}(x) - 2x^2\epsilon_i\Lambda^{2N-2}$ with respect to $x$ at $x = p_i$ does not vanish, there are no such higher order terms $(x-p_i)^{-a}, a = 3, 4, 5, \cdots$ in the effective superpotential (2.5).

The variation of $W_{eff}$ with respect to $B_i$ leads to, by the formula of contour integral,

$$0 = \oint \frac{P_{2N}(x) - 2x^2 \epsilon_i \Lambda^{2N-2}}{(x-p_i)^2} dx = \left( P_{2N}(x) - 2x^2 \epsilon_i \Lambda^{2N-2} \right)' |_{x=p_i}$$

$$
\begin{aligned}
&= \left(P'_{2N}(x) - \frac{2}{x} 2\epsilon_i x^2 \Lambda^{2N-2}\right)|_{x=p_i} = \left(P_{2N}(x) \sum_{J=1}^{N} \frac{2x}{x^2-\phi_J^2} - \frac{2}{x} P_{2N}(x)\right)|_{x=p_i} \\
&= P_{2N}(x)\left(\mathrm{Tr}\frac{1}{x-\Phi_{cl}} - \frac{2}{x}\right)|_{x=p_i} \qquad (2.6)
\end{aligned}
$$

where we used the equation of motion for $L_i$ when we replace $2x^2\epsilon_i\Lambda^{2N-2}$ with $P_{2N}(x)$ at $x=p_i$. The last equality comes from the following relation, together with (2.2),

$$
\begin{aligned}
\mathrm{Tr}\frac{1}{x-\Phi_{cl}} &= \sum_{k=0}^{\infty} x^{-k-1}\mathrm{Tr}\Phi_{cl}^{k} = \sum_{i=0}^{\infty} x^{-(2i+1)}\mathrm{Tr}\Phi_{cl}^{2i} = \sum_{i=0}^{\infty} x\left(x^2\right)^{-(i+1)} \sum_{I=1}^{N} 2\left(\phi_I^2\right)^{i} \\
&= \sum_{I=1}^{N} \frac{2x}{x^2-\phi_I^2} \qquad (2.7)
\end{aligned}
$$

where $\Phi_{cl}$ is antisymmetric matrix, the odd power terms are vanishing. Since $P_{2N}(x=p_i) \neq 0$ for $i=1,2,\cdots,(N-n-1)$ due to the relation (2.3) and $H_{2N-2n-2}(x=p_i)=0$, we arrive at, from (2.6),

$$
\left(\mathrm{Tr}\frac{1}{x-\Phi_{cl}} - \frac{2}{x}\right)|_{x=p_i} = 0, \qquad P_{2N}(x=p_0)=0.
$$

Note the presence of $2/x$ term which was not present in the $U(N)$ case. This implies one can get and solve $p_i$ in terms of $u_{2r}$.

As we have used the calculation in (2.6), the derivative $P_{2N}(x)$ with respect to $x$ is given by

$$
P'_{2N}(x) = \left(\prod_{I}^{N}(x^2-\phi_I^2)\right)' = 2x\sum_{J=1}^{N}\prod_{I\neq J}^{N}(x^2-\phi_I^2) = P_{2N}(x)\sum_{J=1}^{N}\frac{2x}{x^2-\phi_J^2}.
$$

Taking into account (2.7), we can rewrite this result as,

$$
\frac{P'_{2N}(x)}{P_{2N}(x)} = \mathrm{Tr}\frac{1}{x-\Phi_{cl}}.
$$

Using (2.7), we have

$$
\frac{P'_{2N}(x)}{P_{2N}(x)} = \sum_{i=0}^{\infty} x^{-(2i+1)}\mathrm{Tr}\Phi_{cl}^{2i} = \sum_{i=0}^{\infty} x^{-(2i+1)} 2i u_{2i}.
$$

After an integration in $x$, we obtain the following result,

$$
P_{2N}(x) = \left[x^{2N}\exp\left(-\sum_{r=1}^{\infty}\frac{u_{2r}}{x^{2r}}\right)\right]_{+} \qquad (2.8)
$$

where we choose an integration constant to zero. This enables us to write $u_{2r}$ with $2r > 2N$ in terms of $u_{2r}$ with $2r \leq 2N$ by requiring the vanishing of negative powers of $x$. The derivative of $P_{2N}(x)$ with respect to $u_{2r}$ is given by

$$
\frac{\partial P_{2N}(x)}{\partial u_{2r}} = -\left[\frac{P_{2N}(x)}{x^{2r}}\right]_{+}
$$

where + means the polynomial part of a Laurent series of the expression inside the bracket. Although we get $W_{low} = W_{low}(g_{2r}, \Lambda)$ by minimizing $W_{eff} = W_{eff}(u_{2r}, p_i, L_i, B_i; g_{2r}, \Lambda)$, we would like to know the information about $F_{2(2n+1)}(x)$ at the minimum by looking at the field equations.

Next we consider variations of $W_{eff}$ with respect to $p_j$,

$$\begin{aligned} 0 &= L_j \oint \frac{P_{2N}(x) - 2x^2\epsilon_i\Lambda^{2N-2}}{(x-p_j)^2}dx + 2B_j \oint \frac{P_{2N}(x) - 2x^2\epsilon_i\Lambda^{2N-2}}{(x-p_j)^3}dx \\ &= 2B_j \oint \frac{P_{2N}(x) - 2x^2\epsilon_i\Lambda^{2N-2}}{(x-p_j)^3}dx. \end{aligned}$$

In the last equation we have used the equation of motion for $B_i$ (2.6). In general this integral does not vanish, then we should have $B_i = 0$ because the curve does not have more than cubic roots since $y^2$ contains a polynomial $H^2_{2N-2n-2}(x)$. We consider variations of $W_{eff}$ with respect to $u_{2r}$,

$$0 = g_{2r} - \sum_{i=0}^{2N-2n-2} \oint \left[\frac{P_{2N}}{x^{2r}}\right]_+ \frac{L_i}{x-p_i}dx,$$

where we used $B_i = 0$ at the level of equation of motion. Multiplying this with $z^{2r-1}$ and summing over $r$ where $z$ is inside the contour of integration we can obtain the following relation,

$$W'(z) \;=\; \sum_{r=1}^{k+1} g_{2r}z^{2r-1} = \sum_{i=0}^{2N-2n-2} \oint \sum_{r=1}^{k+1} z^{2r-1}\frac{P_{2N}(x)}{x^{2r}}\frac{L_i}{(x-p_i)}dx. \tag{2.9}$$

Let us introduce a new polynomial $Q(x)$ defined as

$$\begin{aligned} \sum_{i=0}^{2N-2n-2} \frac{xL_i}{(x-p_i)} &= L_0 + \sum_{i=1}^{N-n-1} xL_i\left(\frac{1}{x-p_i} + \frac{1}{x+p_i}\right) = L_0 + \sum_{i=1}^{N-n-1} \frac{2x^2L_i}{x^2-p_i^2} \\ &\equiv \frac{Q(x)}{H_{2N-2n-2}(x)}. \end{aligned} \tag{2.10}$$

This determines the degree of $Q(x)$ is equal to $(2N-2n-2)$ which is greater than and equal to $(2k-2n)$. Using this new function we can rewrite (2.9) as

$$W'(z) = \oint \sum_{r=1}^{k+1} \frac{z^{2r-1}}{x^{2r}} \frac{Q(x)P_{2N}(x)}{xH_{2N-2n-2}(x)}dx.$$

Since $W'(z)$ is a polynomial of degree $(2k+1)$, we found the order of $Q(x)$ as $(2k-2n)$, so we denote it by $Q_{2k-2n}(x)$. Thus we have found the order of polynomial $Q(x)$ and therefore the order of integrand in (2.9) is like as $\mathcal{O}(x^{2k-(2r+1)})$. Thus if $r \geq k+1$ it does not contribute to the integral because the power of $x$ in this region implies that the Laurent expansion around the origin vanishes. We can replace the upper value of summation with the infinity.

$$W'(z) = \oint \sum_{r=1}^{\infty} \frac{z^{2r-1}}{x^{2r}} \frac{Q_{2k-2n}(x)P_{2N}(x)}{xH_{2N-2n-2}(x)}dx = \oint z\frac{Q_{2k-2n}(x)P_{2N}(x)}{x(x^2-z^2)H_{2N-2n-2}(x)}dx. \tag{2.11}$$

From (2.3) we have the relation,

$$P_{2N}(x) = x\sqrt{F_{2(2n+1)}(x)}H_{2N-2n-2}(x) + \mathcal{O}(x^{-2N+4}).$$

As we substitute this relation to (2.11), the $\mathcal{O}(x^{-2N+4})$ terms do not contribute the integral because the contribution of the Laurent expansion around the origin for the power of $x$ vanishes, so we can drop those terms. Therefore we have

$$W'(z) = \oint z\frac{y_m(x)}{x^2-z^2}dx, \qquad y_m^2(x) = F_{2(2n+1)}(x)Q_{2k-2n}^2(x)$$

corresponding to the equation of motion for the matrix model and the matrix model curve respectively. Then we get an expected and generalized result,

$$\begin{aligned} y_m^2(x) &= F_{2(2n+1)}(x)Q_{2k-2n}^2(x) = W'^2_{2k+1}(x) + \mathcal{O}(x^{2k}) \\ &= W'^2_{2k+1}(x) + f_{2k}(x), \qquad 2n+2 \leq 2k+2 \leq 2N \end{aligned}$$

where both $F_{2(2n+1)}(x)$ and $Q_{2k-2n}(x)$ are functions of $x^2$, then $f_{2k}(x)$ also a function of $x^2$. We put the subscript $m$ in the $y_m$ in order to emphasize the fact that this corresponds to the matrix model curve. When $2k+1 = 2n+1$, we reproduce (2.4) with $Q_0 = g_{2n+2}$. The above relation determines a polynomial $F_{2(2n+1)}(x)$ in terms of $(2n+1)$ unknown parameters by assuming the leading coefficient to be normalized by 1 by assuming that $W(x)$ is known. These parameters can be obtained from both $P_{2N}(x)$ and $H_{2N-2n-2}(x)$ through the factorization condition (2.3). Now we move on for the case of large $k$ next section.

• **Superpotential of degree $2(k+1)$ where $k$ is arbitrary large**

So far we restricted to the case where the degree of superpotential, $2(k+1)$, is less than $2N$. Let us extend the study of previous discussion to general case by allowing a superpotential of any degree. All the $u_i$'s are independent coordinates on a bigger space with appropriate constraints. For this range of degree of polynomial, we have to consider different constraints due to the instanton corrections [26]. We denote $\frac{1}{2k}\langle \mathrm{Tr}\Phi^{2k}\rangle = U_{2k}$ by capital letter. The quantum mechanical expression corresponding to $\mathrm{Tr}\frac{1}{x-\Phi}$ is given by

$$\left\langle \mathrm{Tr}\frac{1}{x-\Phi}\right\rangle = \frac{2N}{x} + \sum_{k=1}^{\infty}\frac{2kU_{2k}}{x^{2k+1}}.$$

Quantum mechanically we do have

$$\left\langle \mathrm{Tr}\frac{1}{x-\Phi}\right\rangle = \frac{d}{dx}\log\left(P_{2N}(x) + \sqrt{P_{2N}^2(x) - 4x^4\Lambda^{4N-4}}\right). \tag{2.12}$$

By integrating over $x$ and exponentiating, we get

$$Cx^{2N}\exp\left(-\sum_{i=1}^{\infty}\frac{U_{2i}}{x^{2i}}\right) = P_{2N}(x) + \sqrt{P_{2N}^2(x) - 4x^4\Lambda^{4N-4}} \tag{2.13}$$

where $C$ is an integration constant that can be determined by the semiclassical limit $\Lambda \to 0$:

$$C x^{2N} \exp\left(-\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) = 2P_{2N}(x) \longrightarrow C = 2.$$

Solving (2.13) with respect to $P_{2N}(x)$,

$$P_{2N}(x) = x^{2N} \exp\left(-\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) + \frac{\Lambda^{4N-4}}{x^{2N-4}} \exp\left(\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right), \tag{2.14}$$

corresponding to (2.8). Since $P_{2N}(x)$ is a polynomial in $x$, (2.14) can be used to express $U_{2r}$ with $2r > 2N$ in terms of $U_{2r}$ with $2r \leq 2N$ by imposing the vanishing of the negative powers of $x$. Let us introduce a new polynomial whose coefficients are Lagrange multipliers. The generalized superpotential with these constraints is described as

$$\begin{aligned} W_{eff} &= \sum_{r=1}^{k+1} g_{2r} U_{2r} + \sum_{i=0}^{2N-2n-2} \left[ L_i \oint \frac{P_{2N}(x) - 2\epsilon_i x^2 \Lambda^{2N-2}}{(x-p_i)} dx + B_i \oint \frac{P_{2N}(x) - 2\epsilon_i x^2 \Lambda^{2N-2}}{(x-p_i)^2} dx \right] \\ &+ \oint R_{2k-2N+2}(x) \left[ x^{2N} \exp\left(-\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) + \frac{\Lambda^{4N-4}}{x^{2N-4}} \exp\left(\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) \right] dx \end{aligned}$$

where $R_{2k-2N+2}(x)$ is a polynomial of degree $(2k - 2N + 2)$ whose coefficients are regarded as Lagrange multipliers which impose constraints $U_{2r}$ with $2r > 2N$ in terms of $U_{2r}$ with $2r \leq 2N$. This expression is a generalization of (2.5).

Now we follow the previous method. The derivative of $W_{eff}$ with respect to $U_{2r}$ leads to

$$\begin{aligned} 0 &= g_{2r} + \oint \frac{R_{2k-2N+2}(x)}{x^{2r}} \left( -x^{2N} \exp\left(-\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) + \frac{\Lambda^{4N-4}}{x^{2N-4}} \exp\left(\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) \right) dx \\ &+ \oint \sum_{i=0}^{2N-2n-2} \frac{L_i}{(x-p_i)} \frac{\partial P_{2N}(x)}{\partial U_{2r}} dx. \end{aligned} \tag{2.15}$$

Using (2.13) we have the relation,

$$-x^{2N} \exp\left(-\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) + \frac{\Lambda^{4N-4}}{x^{2N-4}} \exp\left(\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) = -\sqrt{P_{2N}^2(x) - 4x^4 \Lambda^{4N-4}}$$

and

$$\frac{\partial P_{2N}(x)}{\partial U_{2r}} = -\frac{P_{2N}(x)}{x^{2r}} \ \text{ for } \ 2r \leq 2N, \qquad \frac{\partial P_{2N}(x)}{\partial U_{2r}} = 0 \ \text{ for } \ 2r > 2N.$$

Using these relations we can rewrite (2.15) and simplify as follows:

$$0 = g_{2r} + \oint \frac{R_{2k-2N+2}(x)}{x^{2r}} \left( -\sqrt{P_{2N}^2(x) - 4x^4 \Lambda^{4N-4}} \right) dx - \oint \sum_{i=0}^{2N-2n-2} \frac{L_i}{(x-p_i)} \frac{P_{2N}(x)}{x^{2r}} dx. \tag{2.16}$$

From the massless monopole constraint (2.3) the characteristic function has the following form,

$$P_{2N}(x) = x H_{2N-2n-2}(x) \sqrt{F_{2(2n+1)}(x)} + \mathcal{O}(x^{-2N+4}). \tag{2.17}$$

Substituting (2.17) and (2.3) into (2.16), the $\mathcal{O}(x^{-2N+4})$ terms do not contribute to integral as we have done before, then one gets the coupling $g_{2r}$ as follows:

$$\begin{aligned} g_{2r} &= \oint \frac{R_{2k-2N+2}(x)}{x^{2r}} x H_{2N-2n-2}(x)\sqrt{F_{2(2n+1)}(x)}dx \\ &+ \oint \sum_{i=0}^{2N-2n-2} \frac{L_i}{(x-p_i)} \frac{xH_{2N-2n-2}(x)\sqrt{F_{2(2n+1)}(x)}}{x^{2r}}dx. \end{aligned}$$

As in previous analysis, by multiplying $z^{2r-1}$ and summing over $r$, we will get

$$\begin{aligned} W'(z) &= \sum_{r=1}^{k+1}\left(\oint \frac{R_{2k-2N+2}(x)}{x^{2r}} x H_{2N-2n-2}(x)\sqrt{F_{2(2n+1)}(x)}dx \right. \\ &+ \left. \oint \sum_{i=0}^{2N-2n-2} \frac{L_i}{(x-p_i)} \frac{xH_{2N-2n-2}(x)\sqrt{F_{2(2n+1)}(x)}}{x^{2r}}dx\right) z^{2r-1} \\ &= \oint \frac{z}{x^2-z^2}\sqrt{F_{2(2n+1)}(x)}\left[R_{2k-2N+2}(x)H_{2N-2n-2}(x)+R_{2N-2n-2}(x)\right]dx. \end{aligned}$$

Finally we obtain an expected relation of equations of motion for the general order tree level superpotential case,

$$W'(z) = \oint \frac{zy_m}{x^2-z^2}dx$$

if the matrix model curve is given by

$$\begin{aligned} y_m^2 &= F_{2(2n+1)}(x)\widetilde{Q}_{2k-2n}^2(x) \\ &\equiv F_{2(2n+1)}(x)\left(R_{2k-2N+2}(x)H_{2N-2n-2}(x)+R_{2N-2n-2}(x)\right)^2. \end{aligned}$$

When $2n = 2N-2$ (no massless monopoles), $\widetilde{Q}_{2k-2N+2}(x) = R_{2k-2N+2}(x)$. When the degree of superpotential is equal to $2N$, in other words, $2k = 2N-2$, then $\widetilde{Q}_{2k-2n}(x) = R_0 H_{2N-2n-2}(x) + R_{2N-2n-2}(x)$. In particular, for $2n = 2N-2$, $\widetilde{Q}_0$ is a constant and $y_m^2(x) = F_{4N-2}(x) = x^{-2}(P_{2N}^2(x) - 4\Lambda^{4N-4}x^4)$. The above $y_m^2$ is known up to a polynomial $f_{2k}(x)$ of degree $2k$: $y_m^2 = W'^2_{2k+1}(x) + f_{2k}(x)$.

• **A generalized Konishi anomaly**

Now we are ready to study for a derivation of the generalized Konishi anomaly equation based on the results of previous section. As in [26], we restrict to the case with $\langle \mathrm{Tr} W'(\Phi)\rangle = \mathrm{Tr} W'(\Phi_{cl})$ and assume that the degree of superpotential $(2k+2)$ is less than $2N$. By substituting (2.10) into (2.11) we can write the derivative of superpotential $W'(\phi_I)$

$$W'(\phi_I) = \sum_{i=0}^{2N-2n-2} \oint \phi_I \frac{P_{2N}(x)}{(x^2-\phi_I^2)} \frac{L_i}{(x-p_i)}dx \qquad (2.18)$$

where we varied $W(\phi_I)$ with respect to $\phi_I$ rather than $u_{2r}$ and used the result of $B_i = 0$. Note that $P_{2N}(x) = \prod_{I=1}^{N}(x^2 - \phi_I^2)$. Using this equation, one obtains the following relation,

$$
\begin{aligned}
\mathrm{Tr}\frac{W'(\Phi_{cl})}{z-\Phi_{cl}} &= \mathrm{Tr}\sum_{k=0}^{\infty} z^{-k-1}\Phi_{cl}^k W'(\Phi_{cl}) = \sum_{i=0}^{\infty} z^{-(2i+1)-1} 2\sum_{I=1}^{N}\phi_I^{2i+1}W'(\phi_I) \\
&= 2\sum_{I=1}^{N}\phi_I W'(\phi_I)\frac{1}{(z^2-\phi_I^2)} \\
&= \sum_{I=1}^{N}\frac{2\phi_I^2}{(z^2-\phi_I^2)}\sum_{i=0}^{2N-2n-2}\oint\frac{P_{2N}(x)}{(x^2-\phi_I^2)}\frac{L_i}{(x-p_i)}dx,
\end{aligned}
\tag{2.19}
$$

where when we change the summation index from $k$ to $i$, the only odd terms appear because effectively the product of $\Phi_{cl}$ and $W'(\Phi_{cl})$ does contribute only under that condition. The even terms do not contribute. Here $z$ is outside the contour of integration. In order to compute the above expression we recognize that the following factor can be written as, by simple manipulation between the property of the trace we have seen before,

$$
\sum_{I=1}^{N}\frac{2\phi_I^2}{(z^2-\phi_I^2)(x^2-\phi_I^2)} = \frac{1}{(x^2-z^2)}\left(z\mathrm{Tr}\frac{1}{z-\Phi_{cl}} - x\mathrm{Tr}\frac{1}{x-\Phi_{cl}}\right).
$$

Thus we can write (2.19) as

$$
\mathrm{Tr}\frac{W'(\Phi_{cl})}{z-\Phi_{cl}} = \oint\sum_{i=0}^{2N-2n-2}\frac{P_{2N}(x)L_i}{(x^2-z^2)(x-p_i)}\left(z\mathrm{Tr}\frac{1}{z-\Phi_{cl}} - x\mathrm{Tr}\frac{1}{x-\Phi_{cl}}\right)dx. \tag{2.20}
$$

As in the case of [26], we can rewrite outside contour integral in terms of two parts as follows:

$$
\oint_{z_{out}} = \oint_{z_{in}} - \oint_{C_z + C_{-z}} \tag{2.21}
$$

where $C_z$ and $C_{-z}$ are small contour around $z$ and $-z$ respectively. Thus the first term in (2.20) (corresponding to the second term in $(B.3)$ of [26]) can be written as

$$
\mathrm{Tr}\frac{1}{z-\Phi_{cl}}\oint_{z_{out}}\frac{zQ_{2k-2n}(x)P_{2N}(x)}{xH_{2N-2n-2}(x)(x^2-z^2)}dx
$$

by using the relation (2.10). Let us emphasize that in this case, we cannot drop the terms of order $\mathcal{O}(x^{-2N+4})$. In order to deal with this, we have to use the above change of integration, then this is given by

$$
\begin{aligned}
&\mathrm{Tr}\frac{1}{z-\Phi_{cl}}\left(\oint_{z_{in}}\frac{zQ_{2k-2n}(x)P_{2N}(x)}{xH_{2N-2n-2}(x)(x^2-z^2)}dx - \oint_{C_z+C_{-z}}\frac{zQ_{2k-2n}(x)P_{2N}(x)}{xH_{2N-2n-2}(x)(x^2-z^2)}dx\right) \\
&= \mathrm{Tr}\frac{1}{z-\Phi_{cl}}\left(W'(z) - \frac{y_m(z)P_{2N}(z)}{\sqrt{P_{2N}^2(z) - 4z^4\Lambda^{4N-4}}}\right),
\end{aligned}
$$

where the first term was obtained by the method done previously and in the last equality we used (2.11) and

$$H_{2N-2n-2}(z)=\frac{\sqrt{P_{2N}^2(z)-4\Lambda^{4N-4}z^4}}{z\sqrt{F_{2(2n+1)}(z)}}, \qquad y_m^2(z)=F_{2(2n+1)}(z)Q_{2k-2n}^2(z).$$

The second term was calculated at the poles. The crucial difference between $U(N)$ case and $SO(2N)$ case comes from the second term of (2.20) (corresponding to the first term in $(B.3)$ of [26]), which vanishes in $U(N)$ case. Let us write it as, after an integration over $x$,

$$-\sum_{i=0}^{2N-2n-2}\oint\frac{L_iP_{2N}(x)x}{(x-p_i)(x^2-z^2)}\mathrm{Tr}\frac{1}{x-\Phi_{cl}}dx=-\sum_{i=0}^{2N-2n-2}\frac{L_ip_iP_{2N}(x=p_i)}{(p_i^2-z^2)}\mathrm{Tr}\frac{1}{p_i-\Phi_{cl}}.$$

Now we use the result of the equation of motion for $B_i$ (2.6) in order to change the trace part and arrive at the final contribution as follows:

$$-\sum_{i=0}^{2N-2n-2}2\frac{L_iP_{2N}(x=p_i)}{(p_i^2-z^2)}=-\sum_{i=0}^{2N-2n-2}\oint\frac{2P_{2N}(x)}{(x^2-z^2)}\frac{L_i}{(x-p_i)}dx$$

where note that $z$ is *outside* the contour of integration in last equation. Therefore taking into account for (2.21) and (2.18) we can rewrite the last equation as follows:

$$-2\frac{W'(z)}{z}+\sum_{i=0}^{2N-2n-2}\oint_{C_z+C_{-z}}\frac{2P_{2N}(x)}{(x^2-z^2)}\frac{L_i}{(x-p_i)}dx=-2\frac{W'(z)}{z}+\frac{2}{z}\frac{y_mP_{2N}(z)}{\sqrt{P_{2N}^2(z)-4z^4\Lambda^{4N-4}}}$$

where once again we substituted (2.10) into (2.11). Therefore we obtain

$$\mathrm{Tr}\frac{W'(\Phi_{cl})}{z-\Phi_{cl}}=\mathrm{Tr}\frac{1}{z-\Phi_{cl}}\left(W'(z)-\frac{y_m(z)P_{2N}(z)}{\sqrt{P_{2N}^2(z)-4z^4\Lambda^{4N-4}}}\right)-2\frac{W'(z)}{z}+\frac{2}{z}\frac{y_mP_{2N}(z)}{\sqrt{P_{2N}^2(z)-4z^4\Lambda^{4N-4}}}.$$

Remembering that

$$\mathrm{Tr}\frac{1}{z-\Phi_{cl}}=\frac{P'_{2N}(z)}{P_{2N}(z)},$$

the second and fourth terms are rewritten as follows;

$$-\mathrm{Tr}\frac{1}{z-\Phi_{cl}}\frac{y_m(z)P_{2N}(z)}{\sqrt{P_{2N}^2(z)-4z^4\Lambda^{4N-4}}}+\frac{2}{z}\frac{y_mP_{2N}(z)}{\sqrt{P_{2N}^2(z)-4z^4\Lambda^{4N-4}}}=\frac{2y_m}{z}-\left\langle\mathrm{Tr}\frac{y_m}{z-\Phi}\right\rangle$$

where we used (2.12).

Taking into account the relation,

$$\mathrm{Tr}\frac{W'(\Phi_{cl})-W'(z)}{z-\Phi_{cl}}=\left\langle\mathrm{Tr}\frac{W'(\Phi)-W'(z)}{z-\Phi}\right\rangle$$

we can write as follows:

$$\left\langle\mathrm{Tr}\frac{W'(\Phi)}{z-\Phi}\right\rangle \quad = \quad \left(\left\langle\mathrm{Tr}\frac{1}{z-\Phi}\right\rangle-\frac{2}{z}\right)[W'(z)-y_m(z)]$$

which is the generalized Konishi anomaly equation for $SO(2N)$ case. The resolvent of the matrix model $R(z)$ is related to $W'(z)-y_m(z)$.

## 2.2 Strong gauge coupling approach: $SO(2N+1)$ case

As in $SO(2N)$ case, the perturbed superpotential we are considering is given by (2.1) and the antisymmetric matrix takes the form

$$\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathrm{diag}(i\phi_1, \cdots, i\phi_N, 0).$$

Note that 0 element in the above. The corresponding $\mathcal{N} = 2$ curve is characterized by (See also [84])

$$y^2 = P_{2N}^2(x) - 4\Lambda^{4N-2}x^2 = x^2 H_{2N-2n-2}^2(x) F_{2(2n+1)}(x), \tag{2.22}$$

where the polynomials $H_{2N-2n-2}(x)$ and $F_{2(2n+1)}(x)$ are defined as before. The difference appears in the left hand side only: the power of $\Lambda$ and $x$. In this case also, the solutions for $F_{2(2n+1)}(x)$ are given by (2.4) when the degree of $W'$, $(2k+1)$, is equal to $(2n+1)$. Now one can generalize for different range of the degree of superpotential.

• **Superpotential of degree $2(k+1)$ less than $2N$**

The superpotential with appropriate constraints (2.22) can be written, by noting the power behavior of $x$ and $\Lambda$, as follows:

$$\begin{aligned} W_{eff} &= \sum_{r=1}^{k+1} g_{2r} u_{2r} \\ &+ \sum_{i=0}^{2N-2n-2} \left[ L_i \oint \frac{P_{2N}(x) - 2\epsilon_i x \Lambda^{2N-1}}{(x-p_i)} dx + B_i \oint \frac{P_{2N}(x) - 2\epsilon_i x \Lambda^{2N-1}}{(x-p_i)^2} dx \right]. \end{aligned}$$

All the arguments discussed before hold for $SO(2N+1)$ case and the constraints become

$$\left(P_{2N}(x) - 2x\epsilon_i\Lambda^{2N-1}\right)|_{x=p_i} = 0, \qquad \frac{\partial}{\partial x}\left(P_{2N}(x) - 2x\epsilon_i\Lambda^{2N-1}\right)|_{x=p_i} = 0.$$

Note the behavior of $\Lambda$ term which is linear in $x$.

The variation of $W_{eff}$ with respect to $B_i$ leads to

$$\begin{aligned} 0 &= \oint \frac{P_{2N}(x) - 2x\epsilon_i\Lambda^{2N-1}}{(x-p_i)^2} dx = \left(P_{2N}(x) - 2x\epsilon_i\Lambda^{2N-1}\right)' |_{x=p_i} \\ &= \left(P'_{2N}(x) - \frac{1}{x} 2\epsilon_i x \Lambda^{2N-1}\right)|_{x=p_i} = \left(P_{2N}(x) \sum_{J=1}^{N} \frac{2x}{x^2 - \phi_J^2} - \frac{1}{x} P_{2N}(x)\right)|_{x=p_i} \\ &= P_{2N}(x) \left(\mathrm{Tr}\frac{1}{x - \Phi} - \frac{1}{x}\right)|_{x=p_i} \end{aligned} \tag{2.23}$$

where we used the equation of motion for $L_i$ when we replace $2x\epsilon_i\Lambda^{2N-1}$ with $P_{2N}(x)$ at $x = p_i$. Since $P_{2N}(x = p_i) \neq 0$ due to the relation (2.22) and $H_{2N-2n-2}(x = p_i) = 0$, we arrive at, from (2.23),

$$\left(\mathrm{Tr}\frac{1}{x-\Phi} - \frac{1}{x}\right)|_{x=p_i} = 0, \qquad P_{2N}(x = p_0) = 0.$$

Note the presence of $1/x$ term which was not present in the $U(N)$ case and also the coefficient 1 is different from 2 in the case of $SO(2N)$ we have considered before. The properties of the characteristic function $P_{2N}(x)$ hold. The variation of $W_{eff}$ with respect to $p_j$ can be obtained and the expression looks similar except the power of $x$ and $\Lambda$. That is,

$$0 = 2B_j \oint \frac{P_{2N}(x) - 2x\epsilon_i \Lambda^{2N-1}}{(x-p_j)^3} dx$$

together with the equation of motion for $B_i$ (2.23). This implies the vanishing of $B_i$ since the integration over $x$ does not vanish in general, according to the same reason as the one in $SO(2N)$ case. The variation of $W_{eff}$ with respect to $u_{2r}$ can be done and from the expression for $g_{2r}$ we can construct $W'(z)$. By considering the correct change of the upper bound in the summation we arrive at the same expression (2.11). The contribution of the terms $\mathcal{O}(x^{-2N+2})$ does not appear in the evaluation of an integration and therefore one obtains

$$W'(z) = \oint z \frac{y_m(x)}{x^2 - z^2} dx, \qquad y_m^2(x) = F_{2(2n+1)}(x) Q_{2k-2n}^2(x)$$

corresponding to the equation of motion for the matrix model and the matrix model curve respectively. Then we get a generalized result,

$$y_m^2(x) = F_{2(2n+1)}(x) Q_{2k-2n}^2(x) = {W'}^2_{2k+1}(x) + \mathcal{O}(x^{2k}) = {W'}^2_{2k+1}(x) + f_{2k}(x)$$

where both $F_{2(2n+1)}(x)$ and $Q_{2k-2n}(x)$ are functions of $x^2$, then $f_{2k}(x)$ also a function of $x^2$.

- **Superpotential of degree $2(k+1)$ where $k$ is arbitrary large**

Quantum mechanically we have

$$\left\langle \mathrm{Tr} \frac{1}{x - \Phi} \right\rangle = \frac{d}{dx} \log \left( P_{2N}(x) + \sqrt{P_{2N}^2(x) - 4x^2 \Lambda^{4N-2}} \right). \tag{2.24}$$

Solving this with respect to $P_{2N}(x)$, one gets, with different power behavior in the second term below,

$$P_{2N}(x) = x^{2N} \exp\left( - \sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}} \right) + \frac{\Lambda^{4N-2}}{x^{2N-2}} \exp\left( \sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}} \right).$$

Let us introduce a new polynomial whose coefficients are Lagrange multipliers as usual. The superpotential with these constraints is described as

$$\begin{aligned} W_{eff} &= \sum_{r=1}^{k+1} g_{2r} U_{2r} + \sum_{i=0}^{2N-2n-2} \left[ L_i \oint \frac{P_{2N}(x) - 2\epsilon_i x \Lambda^{2N-1}}{(x-p_i)} dx + B_i \oint \frac{P_{2N}(x) - 2\epsilon_i x \Lambda^{2N-1}}{(x-p_i)^2} dx \right] \\ &+ \oint R_{2k-2N+2}(x) \left[ x^{2N} \exp\left( - \sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}} \right) + \frac{\Lambda^{4N-2}}{x^{2N-2}} \exp\left( \sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}} \right) \right] dx \end{aligned}$$

where $R_{2k-2N+2}(x)$ is a polynomial of degree $(2k-2N+2)$ whose coefficients are regarded as Lagrange multipliers which impose constraints $U_{2r}$ with $2r > 2N$ in terms of $U_{2r}$ with $2r \leq 2N$.

The derivative of $W_{eff}$ with respect to $U_{2r}$ leads to

$$
\begin{aligned}
0 &= g_{2r} + \oint \frac{R_{2k-2N+2}(x)}{x^{2r}} \left( -x^{2N} \exp\left( -\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}} \right) + \frac{\Lambda^{4N-2}}{x^{2N-2}} \exp\left( \sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}} \right) \right) dx \\
&+ \oint \sum_{i=0}^{2N-2n-2} \frac{L_i}{(x-p_i)} \frac{\partial P_{2N}(x)}{\partial U_{2r}} dx. \qquad (2.25)
\end{aligned}
$$

Therefore, we have the relation,

$$
-x^{2N} \exp\left( -\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}} \right) + \frac{\Lambda^{4N-2}}{x^{2N-2}} \exp\left( \sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}} \right) = -\sqrt{P_{2N}^2(x) - 4x^2 \Lambda^{4N-2}}
$$

and

$$
\frac{\partial P_{2N}(x)}{\partial U_{2r}} = -\frac{P_{2N}(x)}{x^{2r}} \;\; \text{for} \;\; 2r \leq 2N, \qquad \frac{\partial P_{2N}(x)}{\partial U_{2r}} = 0 \;\; \text{for} \;\; 2r > 2N.
$$

Using these relations we can rewrite (2.25) as follows,

$$
0 = g_{2r} + \oint \frac{R_{2k-2N+2}(x)}{x^{2r}} \left( -\sqrt{P_{2N}^2(x) - 4x^2 \Lambda^{4N-2}} \right) dx - \oint \sum_{i=0}^{2N-2n-2} \frac{L_i}{(x-p_i)} \frac{P_{2N}(x)}{x^{2r}} dx.
$$

From the massless monopole constraint (2.22) we have the relation,

$$
P_{2N}(x) = x H_{2N-2n-2}(x) \sqrt{F_{2(2n+1)}(x)} + \mathcal{O}(x^{-2N+2}).
$$

Finally we obtain an expected relation of equations of motion for the general order tree level superpotential case,

$$
W'(z) = \oint \frac{z y_m}{x^2 - z^2} dx
$$

if the matrix model curve is given by

$$
\begin{aligned}
y_m^2 &= F_{2(2n+1)}(x) \widetilde{Q}_{2k-2n}^2(x) \\
&\equiv F_{2(2n+1)}(x) \left( R_{2k-2N+2}(x) H_{2N-2n-2}(x) + R_{2N-2n-2}(x) \right)^2.
\end{aligned}
$$

When $2n = 2N-2$ (no massless monopoles), $\widetilde{Q}_{2k-2N+2}(x) = R_{2k-2N+2}(x)$. When the degree of superpotential is equal to $2N$, in other words, $2k = 2N-2$, then $\widetilde{Q}_{2k-2n}(x) = R_0 H_{2N-2n-2}(x) + R_{2N-2n-2}(x)$. In particular, for $2n = 2N-2$, $\widetilde{Q}_0$ is a constant and $y_m^2(x) = F_{4N-2}(x) = x^{-2}(P_{2N}^2(x) - 4\Lambda^{4N-2}x^2)$.

• **A generalized Konishi anomaly**

Now we are ready to study for a derivation of the generalized Konishi anomaly equation. According to the same change of integration done in previous discussion, $\mathrm{Tr}\frac{W'(\Phi_{cl})}{z-\Phi_{cl}}$ is given by

$$
\begin{aligned}
&\mathrm{Tr}\frac{1}{z-\Phi_{cl}} \left( \oint_{z_{in}} \frac{z Q_{2k-2n}(x) P_{2N}(x)}{x H_{2N-2n-2}(x)(x^2-z^2)} dx - \oint_{C_z + C_{-z}} \frac{z Q_{2k-2n}(x) P_{2N}(x)}{x H_{2N-2n-2}(x)(x^2-z^2)} dx \right) \\
&= \mathrm{Tr}\frac{1}{z-\Phi_{cl}} \left( W'(z) - \frac{y_m(z) P_{2N}(z)}{\sqrt{P_{2N}^2(z) - 4z^2 \Lambda^{4N-2}}} \right),
\end{aligned}
$$

where in the last equality we used (2.11) and

$$H_{2N-2n-2}(z)=\frac{\sqrt{P_{2N}^2(z)-4\Lambda^{4N-2}z^2}}{z\sqrt{F_{2(2n+1)}(z)}},\qquad y_m^2(z)=F_{2(2n+1)}(z)Q_{2k-2n}^2(z).$$

Let us write it as, after an integration over $x$,

$$-\sum_{i=0}^{2N-2n-2}\oint\frac{L_iP_{2N}(x)x}{(x-p_i)(x^2-z^2)}\mathrm{Tr}\frac{1}{x-\Phi_{cl}}dx=-\sum_{i=0}^{2N-2n-2}\frac{L_ip_iP_{2N}(x=p_i)}{p_i^2-z^2}\mathrm{Tr}\frac{1}{p_i-\Phi_{cl}}.$$

Now we use the result of the equation of motion for $B_i$ (2.23) in order to change the trace part and arrive at the final contribution as follows:

$$-\sum_{i=0}^{2N-2n-2}\frac{L_iP_{2N}(x=p_i)}{p_i^2-z^2}=-\sum_{i=0}^{2N-2n-2}\oint\frac{P_{2N}(x)}{(x^2-z^2)}\frac{L_i}{(x-p_i)}dx$$

where note that $z$ is *outside* the contour of integration in last equation. As in $SO(2N)$ case we can rewrite the last equation as follows:

$$-\frac{W'(z)}{z}+\sum_{i=0}^{2N-2n-2}\oint_{C_z+C_{-z}}\frac{P_{2N}(x)}{(x^2-z^2)}\frac{L_i}{(x-p_i)}dx=-\frac{W'(z)}{z}+\frac{1}{z}\frac{y_mP_{2N}(z)}{\sqrt{P_{2N}^2(z)-4z^2\Lambda^{4N-2}}}$$

Therefore we obtain

$$\mathrm{Tr}\frac{W'(\Phi_{cl})}{z-\Phi_{cl}}=\mathrm{Tr}\frac{1}{z-\Phi_{cl}}\left(W'(z)-\frac{y_m(z)P_{2N}(z)}{\sqrt{P_{2N}^2(z)-4z^2\Lambda^{4N-2}}}\right)-\frac{W'(z)}{z}+\frac{1}{z}\frac{y_mP_{2N}(z)}{\sqrt{P_{2N}^2(z)-4z^2\Lambda^{4N-2}}}.$$

Remembering that

$$\mathrm{Tr}\frac{1}{z-\Phi_{cl}}=\frac{P_{2N}'(z)}{P_{2N}(z)},$$

the second and fourth terms are rewritten as follows;

$$-\mathrm{Tr}\frac{1}{z-\Phi_{cl}}\frac{y_m(z)P_{2N}(z)}{\sqrt{P_{2N}^2(z)-4z^2\Lambda^{4N-2}}}+\frac{1}{z}\frac{y_mP_{2N}(z)}{\sqrt{P_{2N}^2(z)-4z^2\Lambda^{4N-2}}}=\frac{y_m}{z}-\left\langle\mathrm{Tr}\frac{y_m}{z-\Phi}\right\rangle$$

where we used (2.24).

We can summarize as follows,

$$\left\langle\mathrm{Tr}\frac{W'(\Phi)}{z-\Phi}\right\rangle\;=\;\left(\left\langle\mathrm{Tr}\frac{1}{z-\Phi}\right\rangle-\frac{1}{z}\right)[W'(z)-y_m(z)]$$

This is the generalized Konishi anomaly equation for $SO(2N+1)$ case.

## 2.3 The multiplication map and the confinement index

In this subsection, we discuss some important property of $\mathcal{N}=1$ $SO(N)$ gauge theories for surveying phase structure. As already discussed, this theory is $\mathcal{N}=2$ theory deformed by tree level superpotential (2.1) with symmetry breaking pattern,

$$SO(N) \to SO(N_0) \times \prod_{i=1}^{n} U(N_i). \tag{2.26}$$

At first let us review some notations and ideas used in [25]. The matrix model curve $\Sigma$, which have already been derived from the strong gauge coupling approach, with $2k=2n$, is characterized by (2.4)

$$y_m^2 = W'_{2n+1}(x)^2 + f_{2n}(x) \tag{2.27}$$

and plays an important role for understanding the various phase structures or the effective superpotential of corresponding gauge theories. At most, there are $(2n+1)$ branch cuts on the curve $\Sigma$. Let us define $A_i$ as a cycle surrounding the $i$-th cut and $B_i$ as a cycle connecting two infinities of the curve $\Sigma$ through $i$-th cut. These cycles are simplectic pair that intersect each other with the intersection number, $(A_i, A_j) = (B_i, B_j) = 0, (A_i, B_j) = \delta_{ij}$. Since the polynomial $f_{2n}(x)$ is an even function in $x$, this Riemann surface $\Sigma$ has $\mathbf{Z}_2$ identification, which is specific to $SO/Sp$ cases. As in [25], let us introduce the two operators,

$$T(x) = \mathrm{Tr}\frac{1}{x-\Phi}, \qquad R(x) = -\frac{1}{32\pi^2}\mathrm{Tr}\frac{W_\alpha W^\alpha}{x-\Phi}.$$

The period integrals of these operators have a very nice interpretation. At first, the period integrals of $R(x)$ lead to

$$\oint_{A_i} R(x)dx = S_i, \qquad \oint_{B_i} R(x)dx = \Pi_i = \frac{1}{2\pi i}\frac{\partial \mathcal{F}}{\partial S_i}$$

where $S_i$ are glueball superfields of the gauge groups $SO(N_0)$ and $U(N_i)$ in the above, and $\mathcal{F}$ is the prepotential of the gauge theory. The others are period integrals of $T(x)$,

$$\oint_{A_i} \overline{T}(x)dx = N_i, \qquad \oint_{B_i} \overline{T}(x)dx = -\tau_0 - b_i$$

where we defined $\tau_0 = -\oint_{B_0} \overline{T}(x)dx$ and introduced $\overline{T}(x)$ and the real number $b_i$ is written as the sum of a period of $T(x)$ around the other compact cycle $C_i$ which is related to the noncompact cycle $B_i$, according to the convention of [26].

Next let us review the effective superpotential by using these quantities. In [22], the perturbative analysis of effective superpotential for $SO/Sp$ gauge theory was discussed from purely field theory viewpoint. The contribution comes from only $\mathbf{S}^2$ and $\mathbf{RP}^2$ graphs. The explicit representation of the effective superpotential is given by [22, 60, 62]

$$W_{eff}(S_i) = (N_0 - 2)\frac{\partial \mathcal{F}}{\partial S_0} + \sum_{i=1}^{n} N_i \frac{\partial \mathcal{F}}{\partial S_i} + 2\pi i \tau_0 \sum_{i=0}^{n} S_i + 2\pi i \sum_{i=1}^{n} b_i S_i. \tag{2.28}$$

The first term comes from $\mathbf{S}^2$ contribution and the second term comes from $\mathbf{RP}^2$ contribution. Note that along the line of [26], we introduced a real number $b_i$ and by including this the theta angle is shifted. For $U(N)$ gauge theories, by using 't Hooft loop and Wilson loop, the effect of $b_i$ was discussed. For any given $N$, there are two kinds of vacua, namely the confining vacua and the Coulomb vacua. The criterion of these vacua is quite simple. If both $N_i$ and $b_i$ have a common divisor, then the vacua are confining vacua. The most greatest common divisor is called a confinement index. If there is no common divisor between them and the vacuum does not have a confinement, then it is called Coulomb vacua: for example, those vacua in which any of the $b_j$ is 1.

Since the number of vacua of $SO(N_0)$ gauge theories is $(N_0 - 2)$ that is the dual Coxeter number of the group, total number of vacua is given by $(N_0 - 2) \times \prod_{i=1}^{n} N_i$. If we introduce a new notations $\hat{N}_0 \equiv N_0 - 2$ and $\hat{N}_i = N_i$, we can discuss the counting of vacua in parallel way as $U(N)$ case. These low energy vacua are characterized by some integers $r_i, (i = 0, \cdots, n)$ with $0 \leq r_i \leq \hat{N}_i - 1$. We can easily extend the criterion of $U(N)$ gauge theories to our $SO(N)$ gauge theories. If both $\hat{N}_i$ and $b_i \equiv r_i - r_{i+1}$ have a common divisor, the vacua are confining vacua. If not, those are Coulomb vacua.

In [26] to probe confinement, the center of $SU(N)$, $\mathbf{Z}_N$, was used. The transformation property of some representation under the center was important for studying Wilson loop $W$. How about $SO/Sp$ gauge theories? Since the centers of $SO/Sp$ gauge theories are $\mathbf{Z}_2$ rather than $\mathbf{Z}_N$, it seems that the confinement index of the theories can be smaller. When we consider some representation $\mathcal{R}$ as a tensor product of $r$ copies of the defining representation, for some $r \geq 0$, there exist two types of transformation under the center. When the $r$ is even, it is invariant under the $\mathbf{Z}_2$ while it transforms by $-1$ when the $r$ is odd. Thus Wilson loop $W$ in $\mathcal{R}$ is defined mod 2 and $W^2$ has no area law (the theory is completely unconfining). Naively if one proceeds the discussion of [26], one regards the confinement index $t$ in the notation of [26] as the greatest common divisor of three quantities 2, $N_i$ and $b_i$. When $t = 2$, there exists only confining phase, in other words, all phases are confining (For $t = 1$ which is trivial, the theory is completely unconfining). This result seems to be unnatural because as we will see later, we explicitly present examples for phase that can not be obtained by multiplication map. One can interpret this phase as Coulom phase from the $U(N)$ results. So the definition of confinement index above is not appropriate and from another mechanism [1] it allows us to introduce the

[1]For $SO(N_0)$ gauge theory, there exist $(N_0-2)$ vacua which can be labeled by $r_0$ with $0 \leq r_0 \leq (N_0-2)-1$.

right definition of confinment index.

Thus we claim that for $SO(N)$ gauge group breaking into $SO(N_0) \times \prod_{i=1}^{n} U(N_i)$, the confinement index $t$ is nothing but the greatest common divisor of $\hat{N}_i$ and $b_i$ which is exactly the same as multiplication map index $K$. As we will see in the various examples of section 2.4, this definition is consistent with the results of explicit calculation for given gauge theories and all the examples we consider support our claim.

• $SO(2N)$ **case**

For any positive integer $K$, the construction maps vacua of the $SO(2N)$ theory with a given superpotential $W(x)$ to vacua of the $SO(2KN-2K+2)$ theory with the *same* superpotential. All the vacua of $SO(2KN-2K+2)$ with confinement index $K$ can be obtained from the Coulomb vacua of $SO(2N)$ under the multiplication map.

Let us assume that $P_{2N}(x)$ satisfies massless monopole constraint with some renormalization scale $\Lambda_0$, as we have seen before (2.3) and (2.4),

$$\begin{aligned} P_{2N}^2(x) - 4x^4\Lambda_0^{4N-4} &= x^2 H_{2N-2n-2}^2(x) F_{2(2n+1)}(x), \\ F_{2(2n+1)}(x) &= W'(x)^2 + f_{2n}(x) \end{aligned}$$

where we put $g_{2n+2} = 1$ and have the symmetry breaking pattern $SO(2N) \to SO(2N_0) \times \prod_{i=1}^{n} U(N_i)$ in the semiclassical limit. The numbers $N_0$ and $N_i$ should satisfy $2N = 2N_0 + 2\sum_{i=1}^{n} N_i$. By using Chebyshev polynomials [2] of the first kind $\mathcal{T}_K(x)$ of degree $K$ and the second kind $\mathcal{U}_{K-1}(x)$ of degree $(K-1)$ [87], we can construct the solution for the massless monopole constraint of $SO(2KN-2K+2)$ with the breaking pattern $SO(2KN_0-2K+2) \times \prod_{i=1}^{n} U(KN_i)$, motivated by [9]

$$P_{2KN-2K+2}(x) = 2\eta^K x^2 \Lambda^{2KN-2K} \mathcal{T}_K\left(\frac{P_{2N}(x)}{2\eta x^2 \Lambda^{2N-2}}\right), \tag{2.29}$$

where $\eta$ is a $2K$-th root of unity, i.e. $\eta^{2K} = 1$. One can check that in the right hand side, the argument of the first kind Chebyshev polynomial has a degree $K(2N-2)$, by the definition of the first kind Chebyshev polynomial and the right hand side has a factor $x^2$, therefore it leads to a polynomial of degree $2 + K(2N-2)$ totally. It is right to write the left hand side as $P_{2KN-2K+2}(x)$ which agrees with the number of order in $x$ in the right hand side. The power

It is natural that one can distinguish each vacuum by the type of confinement as in $U(N)$ case. In other words in each vacuum the loop order parameter $W^p H^q$ are unconfined where $H$ is 'tHooft loop. Which values of $p$ and $q$ are valid for the $r_0$-th vacuum? When we consider the Wilson loop we have only to consider the center because of electric screening explained in [26]. Since the center is $\mathbf{Z}_2$ the Wilson loop $W^2$ seems to have no area law. Thus to distinguish the vacua by the type of confinement, it is necessary to consider the case with $p = 1, 1 \le q \le N_0 - 2$. We conjecture that in the $r_0$-th vacuum, the $W_0 H_0^{r_0}$ are unconfined and from the effect of magnetic screening $H^{N_0-2}$ is unconfined.

[2] These Chebyshev polynomials $\mathcal{T}_K(x)$ and $\mathcal{U}_K(x)$ are slightly different from $T_K(x)$ and $U_K(x)$ used in [9]. These polynomials have the relation, $\frac{1}{2} T_K(2x) = \mathcal{T}_K(x)$ and $U_K(2x) = \mathcal{U}_{K-1}(x)$.

of $\Lambda$ for the argument of Chebychev polynomial was fixed by the power of $x$ which is equal to $(2N-2)$. The power of $\Lambda$ in front of the Chebyshev polynomial can be fixed by the dimension consideration of both sides. The right side should contain $(2KN-2K+2)$ which is equal to $2+(2KN-2K)$ where the first term come from the power of $x$ and the second one should be the power of $\Lambda$. Also note that the same $\eta$ term appears in the denominator of the argument of Chebyshev polynomial. Since Chebyshev polynomials have useful relation,

$$\mathcal{T}_K^2(x)-1=(x^2-1)\mathcal{U}_{K-1}^2(x),$$

we can confirm that (2.29) really satisfies the massless monopole constraint of $SO(2KN-2K+2)$ gauge theories.

For the simplicity of equations, let us define

$$\widetilde{x}\equiv\frac{P_{2N}(x)}{2\eta x^2\Lambda^{2N-2}}.$$

Then one can easily check that

$$\begin{aligned}
P_{2KN-2K+2}^2(x)-4x^4\Lambda^{4KN-4K} &= 4x^4\Lambda^{4KN-4K}\left[\mathcal{T}_K^2\left(\widetilde{x}\right)-1\right]\\
&= 4x^4\Lambda^{4KN-4K}\left(\widetilde{x}^2-1\right)\mathcal{U}_{K-1}^2\left(\widetilde{x}\right)\\
&= \frac{\Lambda^{4KN-4K}}{\eta^2\Lambda^{4N-4}}\left(P_{2N}^2(x)-4x^4\eta^2\Lambda^{4N-4}\right)\mathcal{U}_{K-1}^2\left(\widetilde{x}\right)\\
&= x^2\left[H_{2N-2n-2}(x)\eta^{-1}\Lambda^{2(K-1)(N-1)}\mathcal{U}_{K-1}(\widetilde{x})\right]^2F_{2(2n+1)}(x).
\end{aligned}$$

In the fourth equality we used the identification $\Lambda_0^{4N-4}=\eta^2\Lambda^{4N-4}$. This implies that the following relations

$$\begin{aligned}
P_{2KN-2K+2}(x) &= 2\eta^K x^2\Lambda^{2KN-2K}\mathcal{T}_K\left(\frac{P_{2N}(x)}{2\eta x^2\Lambda^{2N-2}}\right),\\
\widetilde{F}_{2(2n+1)}(x) &= F_{2(2n+1)}(x),\\
H_{(2N-2)K-2n}(x) &= H_{2N-2n-2}(x)\eta^{-1}\Lambda^{2(K-1)(N-1)}\mathcal{U}_{K-1}\left(\frac{P_{2N}(x)}{2\eta x^2\Lambda^{2N-2}}\right)
\end{aligned}$$

satisfy the solution of

$$P_{2KN-2K+2}^2(x)-4x^4\Lambda^{4NK-4K}=x^2H_{(2N-2)K-2n}^2(x)\widetilde{F}_{2(2n+1)}(x).$$

Since $\widetilde{F}_{2(2n+1)}(x)=F_{2(2n+1)}(x)$, the vacua constructed this way for the $SO(2KN-2K+2)$ theory have the *same* superpotential as the vacua of the $SO(2N)$ theory. For a superpotential $W(x)$, the $SO(2N)$ theory has a finite number of vacua with given $n$. For $K$ different values of the $SO(2N)$ parameter $\Lambda_0^{4N-4}$, that is,

$$\Lambda_0^{4N-4}=\eta^2\Lambda^{4N-4}$$

there exists $SO(2KN-2K+2)$ vacua.

Next we consider the multiplication map of $T(x)$ we introduced,

$$\begin{aligned} T(x) &= \frac{d}{dx}\log\left(P_{2N}(x)+\sqrt{P_{2N}^2(x)-4x^4\Lambda^{4N-4}}\right) \\ &= \frac{P'_{2N}(x)}{\sqrt{P_{2N}^2(x)-4x^4\Lambda^{4N-4}}}-\frac{2P_{2N}(x)}{x\sqrt{P_{2N}^2(x)-4x^4\Lambda^{4N-4}}}+\frac{2}{x}. \end{aligned} \tag{2.30}$$

Remember that the Seiberg-Witten differential has the form of

$$d\lambda_{SW}=\frac{xdx}{\sqrt{P_{2N}^2(x)-4x^4\Lambda^{4N-4}}}\left(P'_{2N}(x)-\frac{2}{x}P_{2N}(x)\right).$$

By using (2.29) we obtain the following relations,

$$\begin{aligned} T_K(x) &= \sqrt{P_{2KN-2K+2}^2(x)-4x^4\Lambda^{4KN-4K}} \\ &= \sqrt{\eta^{-2}\Lambda^{4(K-1)(N-1)}\mathcal{U}_{K-1}^2(\widetilde{x})\left(P_{2N}^2(x)-4x^4\Lambda_0^{4N-4}\right)} \\ &= \eta^{K-1}\Lambda^{2(K-1)(N-1)}\mathcal{U}_{K-1}(\widetilde{x})\sqrt{P_{2N}^2(x)-4x^4\Lambda_0^{4N-4}}. \end{aligned} \tag{2.31}$$

In the last equality we used the fact that $\eta^{2K}=1$. That is, $\eta^{-2}=\eta^{2K-2}$. By using the property between the first kind Chebyshev and the second kind of Chebyshev and changing the derivative of the former with respect to the argument into the confinement index $K$ multiplied by the latter, the derivative of $P_{2KN-2K+2}(x)$ with respect to $x$ is given by

$$\begin{aligned} P'_{2KN-2K+2}(x) &= 2\eta^K\Lambda^{2KN-2K}\left[2x\mathcal{T}_K(\widetilde{x})+x^2\left(\frac{P_{2N}(x)}{2\eta x^2\Lambda^{2N-2}}\right)'K\mathcal{U}_{K-1}(\widetilde{x})\right] \\ &= 2\eta^K\Lambda^{2KN-2K}\left[2x\mathcal{T}_K(\widetilde{x})+\frac{P'_{2N}(x)}{2\eta\Lambda^{2N-2}}K\mathcal{U}_{K-1}(\widetilde{x})-\frac{P_{2N}(x)}{\eta x\Lambda^{2N-2}}K\mathcal{U}_{K-1}(\widetilde{x})\right] \\ &= 4x\eta^K\Lambda^{2KN-2K}\mathcal{T}_K(\widetilde{x})+\eta^{K-1}\Lambda^{2(K-1)(N-1)}P'_{2N}(x)K\mathcal{U}_{K-1}(\widetilde{x}) \\ &- \frac{2}{x}\eta^{K-1}\Lambda^{2(K-1)(N-1)}P_{2N}(x)K\mathcal{U}_{K-1}(\widetilde{x}). \end{aligned} \tag{2.32}$$

From these two relations (2.31) and (2.32), we compute the first term of (2.30),

$$\begin{aligned} \frac{P'_{2KN-2K+2}(x)}{\sqrt{P_{2KN-2K+2}^2(x)-4x^4\Lambda^{4KN-4K}}} &= \frac{KP'_{2N}(x)}{\sqrt{P_{2N}^2(x)-4x^4\Lambda_0^{4N-4}}}-\frac{2}{x}\frac{KP_{2N}(x)}{\sqrt{P_{2N}^2(x)-4x^4\Lambda_0^{4N-4}}} \\ &+ 4x\eta\Lambda^{2N-2}\frac{\mathcal{T}_K(\widetilde{x})}{\mathcal{U}_{K-1}(\widetilde{x})\sqrt{P_{2N}^2(x)-4x^4\Lambda_0^{4N-4}}}. \end{aligned} \tag{2.33}$$

The second and third terms of (2.30) become

$$\frac{2}{x}-\frac{2}{x}\frac{P_{2KN-2K+2}(x)}{\sqrt{P_{2KN-2K+2}^2(x)-4x^4\Lambda^{4KN-4K}}}=\frac{2}{x}-\frac{4\eta\Lambda^{2N-2}x\mathcal{T}_K(\widetilde{x})}{\mathcal{U}_{K-1}(\widetilde{x})\sqrt{P_{2N}^2(x)-4x^4\Lambda_0^{4N-4}}}. \tag{2.34}$$

Thus combining (2.33) with (2.34), we obtain the multiplication map of $T(x)$, $T_K(x)$,

$$\begin{aligned} T_K(x) &= K\left(\frac{P'_{2N}(x)}{\sqrt{P^2_{2N}(x)-4x^4\Lambda_0^{4N-4}}}-\frac{2}{x}\frac{P_{2N}(x)}{\sqrt{P^2_{2N}(x)-4x^4\Lambda_0^{4N-4}}}+\frac{2}{x}\right)-K\frac{2}{x}+\frac{2}{x} \\ &= KT(x)-K\frac{2}{x}+\frac{2}{x}. \end{aligned}$$

What does this mean? For the physical meaning, we consider the integral of $T(x)$ around origin,

$$\oint_{x=0} T_K dx = K\oint_{x=0} T dx - 2K + 2 \iff 2N'_0 - 2 = K(2N_0-2).$$

This equation implies that under the multiplication map $2N_0$ does not simply multiply by $K$. The special combination $(2N_0-2)$ have simple multiplication by $K$. This combination is natural for $SO(2N)$ gauge theory because in the effective superpotential (2.28) unoriented diagram gives $-2$ factor.

If we define a new operator $h(x)$ that corresponds to flux one form on Riemann surface $\Sigma$ [11, 12] as $h(x)=T(x)-\frac{2}{x}=\frac{1}{x}\frac{d\lambda_{SW}}{dx}$, we have more simple relation,

$$h_K(x) = Kh(x).$$

After all we can understand that the multiplication map multiplies both $(2N_0-2)$ and $N_i$ by a common $K$. Since these quantities have common devisor $K$ this operation generates only confining vacua. Let us denote adjoint chiral superfield as $\Phi_0$ in the $SO(2N)$ gauge theory. Then through the multiplication map we constructed the following quantum operator in the $SO(2KN-2K+2)$ gauge theory can be obtained by multiplying the confinement index $K$ by the corresponding operator in the $SO(2N)$ gauge theory as follows:

$$\left\langle \mathrm{Tr}\frac{1}{x-\Phi}\right\rangle - \frac{2}{x} = K\left(\left\langle \mathrm{Tr}\frac{1}{x-\Phi_0}\right\rangle - \frac{2}{x}\right)$$

As in $U(N)$ case, all the vacua in $SO(2KN-2K+2)$ with confinement index $K$ arise in this way from the $SO(2N)$ Coulomb vacua. This can be shown by counting the number of vacua. The multiplication map has $\eta$ that is $2K$-th root of unity. So confining vacua constructed by this map have $K$ times as many vacua as Coulomb vacua. Note that $\Lambda_0^{4N-4}=\eta^2\Lambda^{4N-4}$.

Since these vacua have a confinement index $K$, the three quantities $K(2N_0-2)$, $KN_i$ and $\widetilde{b}_i=\widetilde{r}_i-\widetilde{r}_{i+1}$ have greatest common divisor $K$. As in [26], we can represent such $\widetilde{r}_i$ with $0\le u\le K-1$ as $\widetilde{r}_i=u+Kr_i$. Since there exist $K$ choices of $u$, the number of $SO(2KN-2K+2)$ vacua with confinement index $K$ is indeed $K$ times the corresponding number of $SO(2N)$ Coulomb vacua. Thus we constructed all confining vacua with confinement index $K$ from this map explicitly.

• $SO(2N+1)$ **case**

To begin with let us note some property of Chebyshev polynomial $\mathcal{T}_K(x)$,

$$\mathcal{T}_K(x) = \frac{1}{2}\left[(x+\sqrt{x^2-1})^K + (x-\sqrt{x^2-1})^K\right].$$

If the number $K$ is odd, then $\mathcal{T}_K(x)$ is a odd polynomial in $x$. So the function $x\mathcal{T}_K(x)$ is an even polynomial in $x$. With this in mind let us consider the multiplication map of $SO(2N+1)$ gauge theory. By assumption, we have the characteristic function $P_{2N}(x)$ that satisfies the massless monopole constraint (2.22),

$$P_{2N}^2(x) - 4x^2\Lambda^{4N-2} = x^2 H_{2N-2n-2}^2(x) F_{2(2n+1)}(x)$$

and have the breaking pattern $SO(2N_0+1)\times\prod_{i=1}^n U(N_i)$, where $2N+1 = 2N_0+1+\sum_{i=1}^n 2N_i$. By using this solution we can construct a new solution of $SO(2KN-K+2)$ with breaking pattern $SO(2KN_0-K+2)\times\prod_{i=1}^n U(KN_i)$ in the semiclassical limit,

$$P_{2KN-K+1}(x) = 2\eta^K x\Lambda^{2KN-K}\mathcal{T}_K\left(\frac{P_{2N}(x)}{2\eta x\Lambda^{2N-1}}\right) \tag{2.35}$$

where $K$ is an odd number, so $(2KN-K+1)$ is an even number. One can check that in the right hand side, the argument of the first kind Chebyshev polynomial has a degree $K(2N-1)$ in $x$, by the definition of the first kind Chebyshev polynomial and the right hand side has a factor $x$, therefore it leads to a polynomial of degree $1+K(2N-1)$ totally. It is right to write the left hand side as $P_{2KN-K+1}(x)$ which agrees with the number of order in $x$ in the right hand side. The power of $\Lambda$ for the argument of Chebychev polynomial was fixed by the power of $x$ which is equal to $(2N-1)$. The power of $\Lambda$ in front of the Chebyshev polynomial can be fixed by the dimension consideration of both sides. The right hand side should contain $(2KN-K+1)$ which is equal to $1+(2KN-K)$ where the first term comes from the power of $x$ and the second one should be the power of $\Lambda$. Also note that the same $\eta$ term appears in the denominator of the argument of Chebyshev polynomial.

Let us confirm that this function (2.35) indeed satisfies the massless monopole constraint of $SO(2KN-K+2)$. Using the properties we mentioned, one obtains

$$\begin{aligned}
P_{2KN-K+1}^2(x) - 4x^2\Lambda^{4KN-2K} &= 4x^2\Lambda^{4KN-2K}\left(\mathcal{T}_K^2(\widetilde{x})-1\right) \\
&= 4x^2\Lambda^{4KN-2K}\left(\widetilde{x}^2-1\right)\mathcal{U}_{K-1}^2(\widetilde{x}) \\
&= \frac{\Lambda^{4KN-2K}}{\eta^2\Lambda^{4N-2}}\left(P_{2N}^2(x)-4\eta^2x^2\Lambda^{4N-2}\right)\mathcal{U}_{K-1}^2(\widetilde{x}) \\
&= x^2\left[\eta^{-1}\Lambda^{2(K-1)(2N-1)}H_{2N-2n-2}(x)\mathcal{U}_{K-1}(\widetilde{x})\right]^2 F_{2(2n+1)}(x)
\end{aligned}$$

where $\eta^2\Lambda^{4N-2} = \Lambda_0^{4N-2}$ and this implies that the following identification

$$\begin{aligned} P_{2KN-K+1}(x) &= 2\eta^K x\Lambda^{2KN-2K}\mathcal{T}_K\left(\frac{P_{2N}(x)}{2\eta x\Lambda^{2N-1}}\right), \\ \widetilde{F}_{2(2n+1)}(x) &= F_{2(2n+1)}(x), \\ H_{(2N-1)K-2n-1}(x) &= \eta^{-1}\Lambda^{2(K-1)(2N-1)}H_{2N-2n-2}(x)\mathcal{U}_{K-1}(\widetilde{x}) \end{aligned}$$

leads to the solution of

$$P_{2KN-K+1}^2(x) - 4x^2\Lambda^{4KN-2K} = x^2 H_{(2N-1)K-2n-1}^2(x)F_{2(2n+1)}(x).$$

Since $\widetilde{F}_{2(2n+1)}(x) = F_{2(2n+1)}(x)$ the vacua constructed this way for the $SO(2KN-K+2)$ theory have the *same* superpotential as the vacua of the $SO(2N+1)$ theory.

Let us consider the multiplication map of $T(x)$,

$$\begin{aligned} T(x) &= \frac{d}{dx}\log\left(P_{2N}(x) + \sqrt{P_{2N}^2(x) - 4x^2\Lambda^{4N-2}}\right) \\ &= \frac{P_{2N}'(x)}{\sqrt{P_{2N}^2(x) - 4x^2\Lambda^{4N-2}}} - \frac{P_{2N}(x)}{x\sqrt{P_{2N}^2(x) - 4x^2\Lambda^{4N-2}}} + \frac{1}{x}. \end{aligned}$$

One can easily write $T_K(x)$, from the equation (2.35), as follows:

$$T_K(x) = \eta^{K-1}\Lambda^{(K-1)(2N-1)}\mathcal{U}_{K-1}(\widetilde{x})\sqrt{P_{2N}^2(x) - 4x^2\Lambda_0^{4N-2}}.$$

By using the properties of Chebyshev polynomials, the derivative of $P_{2KN-K+1}(x)$ with respect to $x$ can be summarized by

$$\begin{aligned} P_{2KN-K+1}'(x) &= 2\eta^K\Lambda^{2KN-2K}\mathcal{T}_K(\widetilde{x}) + \eta^{K-1}\Lambda^{(K-1)(2N-1)}P_{2N}'(x)K\mathcal{U}_{K-1}(\widetilde{x}) \\ &- \frac{1}{x}\eta^{K-1}\Lambda^{(K-1)(2N-1)}P_{2N}(x)K\mathcal{U}_{K-1}(\widetilde{x}). \end{aligned}$$

Now it is straightforward to compute the various quantities in $T_K(x)$ and we arrive at and obtain the multiplication map of $T(x)$,

$$\begin{aligned} T_K(x) &= K\left(\frac{P_{2N}'(x)}{\sqrt{P_{2N}^2(x) - 4x^2\Lambda_0^{4N-2}}} - \frac{1}{x}\frac{P_{2N}(x)}{\sqrt{P_{2N}^2(x) - 4x^2\Lambda_0^{4N-2}}} + \frac{1}{x}\right) - K\frac{1}{x} + \frac{1}{x} \\ &= KT(x) - K\frac{1}{x} + \frac{1}{x}. \end{aligned}$$

From this result we obtain

$$\oint_{x=0} T_K dx = K\oint_{x=0} T dx - K + 1 \iff 2N_0' - 1 = K(2N_0 - 1).$$

The following quantum operator in the $SO(2KN-2K+2)$ gauge theory can be obtained by multiplying the confinement index $K$ by the corresponding operator in the $SO(2N+1)$ gauge theory

$$\left\langle \mathrm{Tr}\frac{1}{x-\Phi}\right\rangle - \frac{1}{x} = K\left(\left\langle \mathrm{Tr}\frac{1}{x-\Phi_0}\right\rangle - \frac{1}{x}\right).$$

• **From** $SO(2N+1)$ **to** $SO(2M)$

Are there any multiplication map from $SO(2N)$ to $SO(2M+1)$? Conversely from $SO(2N+1)$ to $SO(2M)$ ? We can construct the multiplication map only from $SO(2N+1)$ to $SO(2M)$ where $2M = 2KN-K+2$. If the number $K$ is even, $\mathcal{T}_K(x)$ is an even function. Let us consider

$$P_{2KN-K+2}(x) = 2\eta^K x^2 \Lambda^{2KN-K} \mathcal{T}_K\left(\frac{P_{2N}(x)}{2\eta x \Lambda^{2N-1}}\right), \quad K \text{ is even}. \tag{2.36}$$

One can check that in the right hand side, the argument of the first kind Chebyshev polynomial has a degree $K(2N-1)$, by the definition of the first kind Chebyshev polynomial and the right hand side has a factor $x^2$, therefore it leads to a polynomial of degree $2+K(2N-1)$ totally. It is right to write the left hand side as $P_{2KN-K+2}(x)$ which agrees with the number of order in $x$ in the right hand side. The power of $\Lambda$ for the argument of Chebychev polynomial was fixed by the power of $x$ which is equal to $(2N-1)$. The power of $\Lambda$ in front of the Chebyshev polynomial can be fixed by the dimension consideration of both sides. The right hand side should contain $(2KN-K+2)$ which is equal to $2+(2KN-K)$ where the first term comes from the power of $x$ and the second one should be the power of $\Lambda$. Also note that the same $\eta$ term appears in the denominator of the argument of Chebyshev polynomial. We can check that this polynomial satisfies the massless monopole constraint of $SO(2KN-K+2)$ explicitly. As we did before

$$\begin{aligned}
P^2_{2KN-K+2}(x) - 4x^4\Lambda^{4KN-2K} &= 4x^4\Lambda^{4KN-2K}\left[\mathcal{T}_K^2(\widetilde{x}) - 1\right] \\
&= 4x^4\Lambda^{4KN-2K}\left(\widetilde{x}^2 - 1\right)\mathcal{U}^2_{K-1}(\widetilde{x}) \\
&= \frac{x^2\Lambda^{4KN-2K}}{\eta^2\Lambda^{4N-2}}\left(P^2_{2N}(x) - 4x^2\eta^2\Lambda^{4N-2}\right)\mathcal{U}^2_{K-1}(\widetilde{x}) \\
&= x^2\left[x\eta^{-1}\Lambda^{2KN-K-2N+1}H_{2N-2n-2}(x)\mathcal{U}_{K-1}(\widetilde{x})\right]^2 F_{2(2n+1)}(x).
\end{aligned}$$

This implies the identification

$$\begin{aligned}
P_{2KN-K+2}(x) &= 2\eta^K x^2\Lambda^{2KN-K}\mathcal{T}_K\left(\frac{P_{2N}(x)}{2\eta x\Lambda^{2N-1}}\right), \\
\widetilde{F}_{2(2n+1)}(x) &= F_{2(2n+1)}(x), \\
H_{(2N-1)K-2n}(x) &= x\eta^{-1}\Lambda^{2KN-K-2N+1}H_{2N-2n-2}(x)\mathcal{U}_{K-1}(\widetilde{x}),
\end{aligned}$$

which lead to the solution of

$$P^2_{2KN-K+2}(x) - 4x^4\Lambda^{4KN-2K} = x^2 H^2_{(2N-1)K-2n}(x) F_{2(2n+1)}(x).$$

As in previous case since $\widetilde{F}_{2(2n+1)}(x) = F_{2(2n+1)}(x)$ the vacua constructed this way for the $SO(2KN-K+2)$ theory have the *same* superpotential as the vacua of the $SO(2N+1)$ theory.

## 2.4 Examples

In this subsection we will deal with and analyze the explicit examples which describe the discussions we did, with rank $n = 1$, namely $SO(2N)$ gauge group is broken to $SO(2N_0) \times U(N_1)$. After the explanation for general $SO(2N)$ and $SO(2N+1)$ gauge theories we will analyze the examples with gauge group $SO(N)$ for $N = 4, 5, 6, 7, 8$. As discussed in multiplication map we can construct a map from $SO(2N+1)$ to $SO(2M)$. So when we survey the confining phase we should note that multiplication map. Then we will consider both $SO(2N)$ and $SO(2N+1)$ in this section. Through these examples we will see that there exists a smooth interpolation between different pairs $(N_0, N_1)$.

$SO(2N)$ **case**

At first we study moduli space for $\mathcal{N} = 1$ $SO(2N)$ gauge theories that is $\mathcal{N} = 2$ theories deformed by tree level superpotential. For the simplicity we consider the special case $k = n = 1$, then the superpotential is

$$W(\Phi) = \frac{m}{2}\mathrm{Tr}\Phi^2 + \frac{g}{4}\mathrm{Tr}\Phi^4.$$

$\mathcal{N} = 2$ theory deformed this potential only has unbroken supersymmetry on submanifolds of the Coulomb branch, where there are additional massless monopoles or dyons. Then we want to find a solution of massless monopole constraint (2.3) with $n = 1$,

$$P^2_{2N}(x) - 4x^4\Lambda^{4N-4} = \left(P_{2N}(x) + 2x^2\Lambda^{2N-2}\right)\left(P_{2N}(x) - 2x^2\Lambda^{2N-2}\right) = x^2 H^2_{2N-4}(x) F_6(x). \quad (2.37)$$

The $\mathcal{N} = 2$ $SO(2N)$ gauge theory whose Coulomb branch was described by this hyperelliptic curve and the $2N$-th order polynomial $P_{2N}(x)$ parametrizes the point in the moduli space. This point is denoted by the $N$ eigenvalues of the matrix $\Phi$. To get $n = 1$, this $\mathcal{N} = 2$ theory should have $(N-2)$ massless magnetic monopoles therefore the polynomial (2.37) should have $2(N-2)$ double roots. Since $P_{2N}(x)$ depends on the $2N$ complex parameters, the subspace on which $P^2_{2N}(x) - 4x^4\Lambda^{4N-4}$ has $2(N-2)$ double roots is two-dimensional. There exist $s_+$ double roots in the first factor and $s_-$ double roots in the second factor. Different values of $s_+$ and $s_-$ correspond to different branches. The right hand side of (2.37) has the factor $x^2$. To have a factor $x^2$ in the left hand side, the constant term in $P_{2N}(x)$ must be zero, that is,

$$P_{2N}(x) = x^{2N} + s_2 x^{2N-2} + s_4 x^{2N-4} + \cdots + s_{2N-2}x^2 + s_{2N}, \quad \text{with} \quad s_{2N} = 0.$$

Thus we obtain the relation $P_{2N}(x) = x^2 P_{2N-2}(x)$. Taking into account this relation, we can rewrite the massless monopole constraint (2.37) as

$$\begin{aligned}
P_{2N}(x) + 2x^2\Lambda^{2N-2} &= x^2 H_{s_+}^2(x) R_{2N-2-2s_+}(x), \\
P_{2N}(x) - 2x^2\Lambda^{2N-2} &= x^2 H_{s_-}^2(x) \widetilde{R}_{2N-2-2s_-}(x), \\
x^2 H_{s_+}(x) H_{s_-}(x) = x H_{2N-4}(x), &\qquad F_6(x) = R_{2N-2-2s_+}(x)\widetilde{R}_{2N-2-2s_-}(x),
\end{aligned} \tag{2.38}$$

where $s_+ + s_- + 1 = 2N - 4$ and $2N - 2 - 2s_\pm \geq 0$. Equivalently it leads to the following factorization problem

$$P_{2N-2}(x) + 2\Lambda^{2N-2} = H_{s_+}^2(x) R_{2N-2-2s_+}(x), \tag{2.39}$$

$$P_{2N-2}(x) - 2\Lambda^{2N-2} = H_{s_-}^2(x) \widetilde{R}_{2N-2-2s_-}(x) \tag{2.40}$$

where the subscripts of the polynomials denote their degrees, respectively. These equations are useful for the later discussion of the relation between $SO(2N)$ case with $Sp(2N)$ case.

From the first two equations (2.38) we obtain

$$H_{s_+}^2(x) R_{2N-2-2s_+}(x) - 4\Lambda^{2N-2} = H_{s_-}^2(x) \widetilde{R}_{2N-2-2s_-}(x). \tag{2.41}$$

This is the relation we will use below for finding the solutions. Then we will study the semi-classical limit $\Lambda \to 0$. We would like to describe for each branch of the moduli space with $N_0$ and $N_1$. Then we break the $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ supersymmetry by turning on a tree level superpotential. Since $n = 1$, the superpotential must have at least three critical points and we consider the case of a quartic superpotential.

As in [26] we introduce the matrix model curve, which give the same Riemann surface $\Sigma$ that appeared in the matrix model context [17],

$$y_m^2 = F_6(x) = R_{2N-2-2s_+}(x)\widetilde{R}_{2N-2-2s_-}(x)$$

and one writes this as

$$R_{2N-2-2s_+}(x)\widetilde{R}_{2N-2-2s_-}(x) = W_3'^2(x) + f_2(x).$$

Since $s_+ + s_- + 1 = 2N - 4$, the left hand side is a 6-th order polynomial and this determines $W_3'(x)$ and hence determines $W(x)$ up to a constant. As already discussed in [11] from the coefficient of $x^2$ in $f_2(x)$ we can find $S = S_0 + S_1$.

$SO(2N+1)$ **case**

We can deal with $SO(2N+1)$ gauge theories in the same way as $SO(2N)$ gauge theories. We consider the same tree level superpotential (2.37) and massless monopole constraint (2.22) with $n = 1$,

$$P_{2N}^2(x) - 4x^2\Lambda^{4N-2} = x^2 H_{2N-4}^2(x) F_6(x).$$

The matrix model curve is determined from the solution of this factorization problem as follows:

$$y_m^2(x) = F_6(x).$$

Now we are ready to deal with the explicit examples.

- $SO(4)$ **case**

The first example we consider is a $SO(4)$ gauge theory. We are considering the $n=1$ case and $2N-2n-2=4-2-2=0$. The characteristic function $P_4(x)$ can be represented as

$$P_4(x) = x^2(x^2-v^2).$$

This is a solution for massless monopole constraint (2.37). In this case massless monopole constraint, the factorization problem, is trivial since we do not need to have double roots at all and by turning on the quartic superpotential it leads to matrix model curve, according to (2.37)

$$y_m^2 = x^{-2}\left(P_4^2(x) - 4x^4\Lambda^4\right) = x^2(x^2-v^2)^2 - 4x^2\Lambda^4.$$

From this curve, we can determine the tree level superpotential and a deformed function $f_2(x)$ as follow,

$$W_3'(x) = x(x^2-v^2), \quad f_2(x) = -4x^2\Lambda^4, \quad S = \Lambda^4.$$

Then there is only one vacuum for a given $W_3'(x)$ that determines $v$ through the above relation. In the semiclassical limit $\Lambda \to 0$, the characteristic function is independent of $\Lambda$ and becomes $P_4(x) \to x^2(x^2-v^2)$. So in this vacuum, the gauge group $SO(4)$ breaks into $SO(2)\times U(1)$.

- $SO(5)$ **case**

The next example is a $SO(5)$ gauge theory. As in previous example, the massless monopole constraint is trivial since we do not need to consider double roots at all. The characteristic function $P_4(x)$ is given by

$$P_4(x) = x^2(x^2-l^2).$$

From this, we obtain the matrix model curve, by turning on the quartic superpotential

$$\begin{aligned}
y_m^2 &= x^{-2}\left(P_4^2(x) - 4x^2\Lambda^6\right) = x^2(x^2-l^2)^2 - 4\Lambda^6, \\
W_3'(x) &= x^2(x^2-l^2), \\
f_2(x) &= -4\Lambda^6.
\end{aligned}$$

There is only one vacuum for given $W_3'(x)$ that determines $l$. The different point compared to $SO(4)$ case is the fact that the sum of glueball superfield is different. As we can see in the above, the function $f_2(x)$ does not have $x$ dependence, therefore the expectation value of glueball superfield is zero, $S=0$. In the semiclassical limit, $P_4(x) \to x^2(x^2-l^2)$, which shows that the gauge group breaks as $SO(5) \to SO(3) \times U(1)$.

- $SO(6)$ **case**

The third example is a $SO(6)$ gauge theory which is more interesting. The massless monopole constraint for this case is given by

$$P_6^2(x) - 4x^4\Lambda^8 = x^2 H_2^2(x) F_6(x). \tag{2.42}$$

If we parametrize $P_6(x) = x^2 P_4(x) = x^2(x^2-a^2)(x^2-b^2)$, the equation (2.42) becomes

$$x^4\left((x^2-a^2)(x^2-b^2)-2\Lambda^4\right)\left((x^2-a^2)(x^2-b^2)+2\Lambda^4\right) = x^2 H_2^2(x) F_6(x).$$

From this equation, the subspace with a massless monopole can be determined by searching for points where $y^2$ has a double roots and we find the relation as follows, by looking at the inside of the bracket in the left hand side and requiring it be a complete square,

$$a^2 - b^2 = \epsilon 2\sqrt{2}\Lambda^4$$

where $\epsilon$ is 4-th root of unity. Thus we have a solution for massless monopole constraint,

$$P_6(x) = x^2(x^2-a^2)(x^2-a^2+2\sqrt{2}\epsilon\Lambda^4).$$

In the classical limit $P_6(x) \to x^2(x^2-a^2)^2$, then the gauge group $SO(6)$ break to $SO(2)\times U(2)$. The matrix model curve by turning on a superpotential which makes a system to put at the point $(a,b)$ in the $\mathcal{N}=2$ moduli space is given by

$$\begin{aligned}
y_m^2 &= x^2\left((x^2-a^2+\epsilon\sqrt{2}\Lambda^4)^2 + 4\epsilon^2\Lambda^8\right), \\
W_3'(x) &= x(x^2-a^2+\epsilon\sqrt{2}\Lambda^4), \\
f_2(x) &= 4x^2\epsilon^2\Lambda^8, \\
S &= -\epsilon^2\Lambda^8.
\end{aligned}$$

These four vacua are confining phase because we can construct multiplication map from $SO(4)$ by $K=2$ where we denote as $P^{K=2}_{SO(4)\to SO(6)}(x)$. According to the analysis of (2.29), $2KN-2K+2$ becomes 6 when $N=2$ and $K=2$ and the characteristic function is given by, where $\mathcal{T}_2(\tilde{x}) = 2\tilde{x}^2-1$,

$$\begin{aligned}
P^{K=2}_{SO(4)\to SO(6)}(x) &= 2\eta^2x^2\Lambda^4\mathcal{T}_2\left(\frac{P_4(x)}{2\eta x^2\Lambda^2}\right) = 2\eta^2x^2\Lambda^4\mathcal{T}_2\left(\frac{x^2(x^2-v^2)}{2\eta x^2\Lambda^2}\right) \\
&= 2\eta^2x^2\Lambda^4\left[2\left(\frac{x^2(x^2-v^2)}{2\eta x^2\Lambda^2}\right)^2 - 1\right] = x^2(x^2-v^2)^2 - 2\eta^2x^2\Lambda^4.
\end{aligned}$$

Then if we choose $v^2 = a^2 - \sqrt{2}\epsilon\Lambda^4$, one gets

$$\begin{aligned} P^{K=2}_{SO(4)\to SO(6)}(x) &= x^2\left((x^2-a^2)^2 + 2\sqrt{2}\epsilon\Lambda^2(x^2-a^2) + 2\Lambda^4(\epsilon^2-\eta^2)\right) \\ &= x^2(x^2-a^2)(x^2-a^2+2\sqrt{2}\epsilon\Lambda^4) = P_6(x) \end{aligned}$$

which we have found before and we used the fact that both $\epsilon$ and $\eta$ are 4-th roots of unity.

Until now we considered confining vacua only. Are there other vacua in $SO(6)$ gauge theory? We can easily see Coulomb vacua for this case as in $SO(4)$, $SO(5)$ gauge theories. If we put $H_2(x) = x^2$ and $P_6(x) = x^2 P_4(x)$, then the massless monopole constraint becomes

$$P_4^2(x) - 4\Lambda^8 = x^2 F_6(x).$$

From this equation we can obtain the solution, $P_4(x) = x^4 + Ax^2 + 2\eta\Lambda^4$ where $\eta$ is 2-th root of unity. In the semiclassical limit, $P_6(x) \to x^4(x^2+A)$, which shows that the gauge group breaks into $SO(6) \to SO(4) \times U(1)$. Thus the matrix model curve is given by

$$\begin{aligned} y_m^2 &= x^2(x^2+A)^2 + 4\eta\Lambda^4(x^2+A), \\ W_3'(x) &= x(x^2+A), \\ f_2(x) &= 4\eta\Lambda^4(x^2+A), \\ S &= -\eta\Lambda^4. \end{aligned}$$

Then there are two vacua for given superpotential $W(x)$.

• $SO(7)$ **case**

Next example is a $SO(7)$ gauge theory which is first example for smooth transition between vacua with different unbroken gauge groups. The massless monopole constraint for this case is given by

$$P_6^2(x) - 4x^2\Lambda^{10} = x^2 H_2^2(x) F_6(x). \tag{2.43}$$

If we appropriately parametrize $P_6(x)$, $H_2(x)$ and $F_6(x)$, the equation (2.43) becomes

$$x^4(x^2-a)^2(x^2-a-b)^2 - 4x^2\Lambda^{10} = x^2(x^2-A)^2\left(x^6 + Bx^4 + Cx^2 + D\right).$$

The solutions of this equation are given by

$$\begin{aligned} B &= -2A + \frac{2\epsilon^2\Lambda^5}{A^{\frac{3}{2}}}, \quad C = A^2 - \frac{6\epsilon^2\Lambda^5}{A^{\frac{1}{2}}} + \frac{\Lambda^{10}}{A^3}, \quad D = -\frac{4\Lambda^{10}}{A^2}, \\ b &= \epsilon\left[\frac{\Lambda^5}{A^{\frac{3}{2}}}\left(8A + \frac{\epsilon^2\Lambda^5}{A^{\frac{3}{2}}}\right)\right]^{\frac{1}{2}}, \quad a = A + \frac{\epsilon^2\Lambda^5}{2A^{\frac{3}{2}}} - \frac{b}{2} \end{aligned} \tag{2.44}$$

where $\epsilon$ is 4-th root of unity. In this case we can take two semiclassical limits with $\Lambda \to 0$:

1. Fixed A: Since $b \to 0$ and $a \to A$, the characteristic function $P_6(x) \to x^2(x^2 - A)^2$. Therefore $SO(7)$ is broken to $SO(3) \times U(2)$. This phase is a Coulomb phase because the numbers $(2N_0 + 1) - 2 = 3 - 2 = 1$ and $N_1 = 2$ that come from unbroken gauge groups have no common divisor.

2. $\Lambda, A \to 0$ with fixed $v \equiv \frac{\epsilon^2\Lambda^5}{A^{\frac{3}{2}}}$: Since $b \to v$ and $a \to 0$, $P_6(x) \to x^4(x^2 - v)$. Therefore $SO(7)$ is broken to $SO(5) \times U(1)$. As previous limit, we have no common divisor of $(2N_0 + 1) - 2 = 5 - 2 = 3$ and $N_1 = 1$. Therefore this phase is a Coulomb phase also. After all we have obtained smooth transition within Coulomb phase.

From the solutions (2.44) we find the corresponding matrix model curve as follows:

$$\begin{aligned} y_m^2 &= x^2\left(x^2 - A + \frac{\epsilon^2\Lambda^5}{A^{\frac{3}{2}}}\right)^2 - \frac{8\epsilon^2\Lambda^5}{A^{\frac{1}{2}}}x^2 - \frac{4\Lambda^{10}}{A^2}, \\ W_3'(x) &= x\left(x^2 - A + \frac{\epsilon^2\Lambda^5}{A^{\frac{3}{2}}}\right), \\ f_2(x) &= -\frac{8\epsilon^2\Lambda^5}{A^{\frac{1}{2}}}x^2 - \frac{4\Lambda^{10}}{A^2}, \\ S &= \frac{2\epsilon^2\Lambda^5}{A^{\frac{1}{2}}}. \end{aligned}$$

Next we count the number of vacua for fixed superpotential, $W_0' = x(x^2 + \Delta)$. How many vacua do we have for this fixed superpotential? From the above result of $W_3'(x)$ we can represent $\Delta$ as

$$\Delta = \frac{\epsilon^2\Lambda^5}{A^{\frac{3}{2}}} - A.$$

We evaluate this equation under the two semiclassical limit 1 and 2 discussed above.

1. In this limit since $\Delta = -A$ we have two functions $f_2(x)$ for each $\epsilon^2$. Therefore we have two vacua. On the other hand, since the unbroken gauge group is $SO(3) \times U(2)$ under this limit, the number of vacua is $2 \cdot 2 = 4$ [3]. This number is not equal to the one derived from the potential. The reason is as follows: We considered only massless monopole constraint (2.22) which have $x^2$ factor. If we include the case without $x^2$ factor, namely, if the branch cut at the origin of the matrix model curve $y_m$ is degenerated, we will have the remaining vacua with the breaking pattern $SO(7) \to SO(3) \times U(2)$ in addition to $SO(7) \to U(3)$ [4].

2. In this limit since $\frac{\epsilon^2}{A^{\frac{1}{2}}} = \left(\frac{\Delta}{\Lambda^5}\right)^{\frac{1}{3}}$, we have three functions $f_2(x)$. Thus there are three vacua. On the other hand, we can count the number of vacua from gauge group. In this limit since the unbroken gauge group is $SO(5) \times U(1)$ the number of vacua is $((2N_0 + 1) - 2) \times N_1 = (5 - 2) \cdot 1 = 3$, which is the same number as the one from potential.

[3]For $SO(N)$ gauge theories with $N \geq 5$ the Witten index is $N - 2$ while for $N \leq 4$ the Witten indices are $1, 2, 4$ for $N = 2, 3, 4$ respectively.

[4]We would like to thank Bo Feng for the discussion on this point.

• $SO(8)$ **case**

At last we study the most interesting example $SO(8)$. We are looking for the subspace of the $\mathcal{N}=2$ gauge theory with $N-n=4-1=3$ monopoles. In this gauge theory since $s_+ + s_- = 3$, we have four branches $(s_+, s_-) = (2,1)$, $(1,2)$ and $(3,0)$, $(0,3)$. The four double roots of the $\mathcal{N}=2$ curve are distributed between the two factors in this region.

**1. Coulomb branch with** $(s_+, s_-) = (2,1)$ **or** $(1,2)$

By introducing $\eta$ we can deal with these two cases in the same way, that is, simultaneously. Taking into account the characteristic function of degree 8, $P_8(x)$ is an even function, we can parametrize points where monopoles are massless as $H_{s_-}(x) = x$ and $H_{s_+}(x) = x^2 - a^2$. The solutions of equations (2.38) or (2.41) are given by

$$\begin{aligned} P_8(x) + 2\eta x^2 \Lambda^6 &= x^2(x^2-a^2)^2(x^2 + \frac{4\eta\Lambda^6}{a^4}), \\ P_8(x) - 2\eta x^2 \Lambda^6 &= x^4 \left[ (x^2-a^2)^2 + \frac{4\eta\Lambda^6}{a^4}(x^2-2a^2) \right]. \end{aligned}$$

Then we find the corresponding matrix model curve as follows:

$$\begin{aligned} y_m^2 &= (x^2 + \frac{4\eta\Lambda^6}{a^4}) \left[ (x^2-a^2)^2 + \frac{4\eta\Lambda^6}{a^4}(x^2-2a^2) \right] \\ &= x^2 \left( x^2 + \frac{4\eta\Lambda^6}{a^4} - a^2 \right)^2 - \frac{8\eta\Lambda^6}{a^2} x^2 + \frac{4\eta\Lambda^6}{a^2}(a^2 - 2\frac{4\eta\Lambda^6}{a^4}) \end{aligned}$$

from which we can see the tree level superpotential and the deformation function $f_2(x)$

$$W_3'(x) = x \left( x^2 + \frac{4\eta\Lambda^6}{a^4} - a^2 \right), \qquad f_2(x) = -\frac{8\eta\Lambda^6}{a^2} x^2 + \frac{4\eta\Lambda^6}{a^2}(a^2 - 2\frac{4\eta\Lambda^6}{a^4}).$$

From the coefficient of $x^2$ in $f_2(x)$, we find $S = \frac{2\eta\Lambda^6}{a^2}$. There are two semiclassical limits with $\Lambda \to 0$,

1. Fixed $a$: The characteristic function behaves $P_8(x) \to x^4(x^2-a^2)^2$. Therefore the gauge group $SO(8)$ is broken to $SO(4) \times U(2)$.

2. $\Lambda, a \to 0$ with fixed $v \equiv \frac{4\eta\Lambda^6}{a^4}$: The characteristic function becomes $P_8(x) \to x^6(x^2+v)$, Therefore $SO(8)$ is broken to $SO(6) \times U(1)$.

These results represent two distinct semiclassical limits in the same moduli space corresponding to different gauge groups. By changing parameter continuously one can freely transit from $SO(4) \times U(2)$ to $SO(6) \times U(1)$. On the other hand, the confining $SO(4) \times U(2)$ vacua below can not make such a transformation because there are no confining $SO(6) \times U(1)$ vacua.

At last we count the number of vacua for fixed tree level superpotential, $W_3'(x) = x(x^2 + \Delta)$. From the previous result of $W_3'(x)$ we can represent $\Delta$ as

$$\Delta = \frac{4\eta\Lambda^6}{a^4} - a^2,$$

in this branch. We evaluate this equation under the two semiclassical limit 1 and 2 discussed above.

1. In this limit since $\Delta = -a^2$ we have two functions $f_2(x)$ for each $\eta$. Thus we have two vacua. As we will see below, we have two vacua in confining phase that have the same gauge group $SO(4) \times U(2)$. Thus all the vacua with this gauge group are four vacua. On the other hand since the gauge group is $SO(4) \times U(2)$ under this limit, the number of vacua is $4 \cdot 2 = 8$. This number is not equal to the one derived from the superpotential analysis. As discussed in $SO(7)$ case if we include the case without $x^2$ factor, namely, if the branch cut at the origin of the matrix model curve $y_m$ is degenerated, we will have the remaining vacua with the breaking pattern $SO(8) \to SO(4) \times U(2)$ in addition to $SO(8) \to U(4)$.

2. In this limit since $a^2 = \left(\frac{2\Lambda^6}{\Delta}\right)^{\frac{1}{2}}$, taking into account for $\eta$ we have four functions $f_2(x)$. In other words we have four vacua for each potential. This number is equal to the one derived from the gauge group $SO(6) \times U(1)$, i.e. $(2N_0 - 2) \times N_1 = (6-2) \cdot 1 = 4$.

**2. Confining branch** $(s_+, s_-) = (0,3)$ **or** $(3,0)$

Next we study the remaining two cases $(s_+, s_-) = (0,3)$ and $(3,0)$. As in previous examples we introduce $\eta = \pm 1$. We can easily obtain a solution for massless monopole constraint,

$$\begin{aligned} P_8(x) + 2\eta x^2 \Lambda^6 &= x^4 (x^2 - a^2)^2 + 4\eta x^2 \Lambda^6, \\ P_8(x) - 2\eta x^2 \Lambda^6 &= x^4 (x^2 - a^2)^2. \end{aligned} \tag{2.45}$$

Then the matrix model curve is given by

$$\begin{aligned} y_m^2 &= x^2 (x^2 - a^2)^2 + 4\eta \Lambda^6, \\ W_3'(x) &= x(x^2 - a^2), \\ f_2(x) &= 4\eta \Lambda^6. \end{aligned}$$

Since the coefficient in front of $x^2$ in $f_2(x)$ is zero, the expectation value of the sum of glueball superfield is zero.

In the semiclassical limit $\Lambda \to 0$, the characteristic function becomes $P_8(x) \to x^4(x^2 - a^2)^2$, which means that $SO(8)$ breaks to $SO(4) \times U(2)$. This breaking pattern has already appeared. However in this case, the phase is a confining phase, which is different from the former.

This solution we have just described is a confining phase because we can construct a multiplication map from $SO(5)$ to $SO(8)$ by $K = 2$ where we denote as $P^{K=2}_{SO(5) \to SO(8)}(x)$. According to the analysis of (2.36), $2KN - K + 2 = 8$ for $N = K = 2$. Plugging these values into (2.36), one gets

$$\begin{aligned} P^{K=2}_{SO(5) \to SO(8)}(x) &= 2\eta^2 x^2 \Lambda^6 \mathcal{T}_2 \left( \frac{P_4(x)}{2\eta x \Lambda^3} \right) = 2\eta^2 x^2 \Lambda^6 \left[ 2 \left( \frac{x^2(x^2 - l^2)}{2\eta x \Lambda^3} \right)^2 - 1 \right] \\ &= x^4 (x^2 - l^2)^2 - 2\eta^2 x^2 \Lambda^6 \end{aligned}$$

where $\eta$ is a 4-th root of unity. If we identify $a^2$ with $l^2$ and taking the $\eta^2$ here as the minus of $\eta$ in (2.45) this equation becomes exactly the solution of (2.45). Thus this vacuum is a confining phase.

So far we have seen the multiplication maps from $SO(4)$ to $SO(6)$ and from $SO(5)$ to $SO(8)$ where the former is an example of the vacua of $SO(2KN-2K+2)$ gauge theory from $SO(2N)$ theory while the latter is an example of the vacua of $SO(2KN-K+2)$ gauge theory where $K$ is even from $SO(2N+1)$ theory. Then it is natural to consider whether there exists the multiplication map from $SO(2N+1)$ to $SO(2KN-K+2)$ where $K$ is odd. For example, when $K=3$ and $N=2$, then the vacua of $SO(11)$ gauge theory has the same superpotential as those of $SO(5)$ theory. Therefore we expect that there exists a solution that is a confining phase from the explicit multiplication map from $SO(5)$ to $SO(11)$ by $K=3$ where we denote as $P^{K=3}_{SO(5)\to SO(11)}(x)$. Although we have considered a couple of examples, it would be interesting to study the possible confining vacua for each general $N$ and $K$ systematically.

One might ask whether one can construct the multiplication map from $SO(2N)$ to $SO(2M+1)$. From the experience we have learned from the section 2.3, one can think of $x\mathcal{T}_K\left(\frac{P_{2N}(x)}{x^2}\right)$ as our characteristic function $P_{2M}(x)$. The reason for linear behavior of $x$ is that as we remember the curve for $SO(2M+1)$ gauge theory contains the $x^2$ in $\Lambda$ term so when we make $P_{2M}(x)$ to be square, we have to have $x^2$ term. Moreover the $1/x^2$ behavior of the first kind Chebyshev polynomial is necessary to require that $SO(2N)$ gauge theory behave like $x^4$ in the $\Lambda$ term and from the relation of first kind Chebyshev and second kind Chebyshev. However, this candidate for the characteristic polynomial has a degree of $1+K(2N-2)$ which is odd, contrary to the fact that in our $SO(2M+1)$ theory, the degree of polynomial should be even. From this naive counting, it does not seem to allow one can construct the multiplication map from $SO(2N)$ to $SO(2M+1)$. Therefore, for $SO(2N)$ and $SO(2N+1)$ gauge theories, there exist only three possible ways to have multiplication maps.

# 3 $Sp(N)$ gauge theory

## 3.1 Strong gauge coupling approach

Let us study the superpotential considered as a small perturbation of an $\mathcal{N}=2$ $Sp(2N)$ gauge theory [85]. The superpotential is given by (2.1) and $\Phi$ is an adjoint scalar chiral superfield and one can transform as follows:

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathrm{diag}(i\phi_1, \cdots, i\phi_N).$$

Note that the trace of odd power of $\Phi$ is zero. We study the vacua in which the perturbation by $W(\Phi)$ (2.1) remains only $U(1)^n$ gauge group at low energies with $2n \leq 2k$. If the remaining degrees of freedom become massive for $W \neq 0$ due to the condensation of $(N-n)$ mutually local magnetic monopoles, then the exact effective superpotential can be written as

$$W_{eff} = \sqrt{2} \sum_{l=1}^{N-n} M_l(u_{2r}) q_l \widetilde{q}_l + \sum_{r=1}^{k+1} g_{2r} u_{2r}$$

similar to the one in $SO(2N)$ case. The mass of monopoles should vanish for $l = 1, 2, \cdots, (N-n)$ in a supersymmetric vacuum.

We consider a singular point in the moduli space where $(N-n)$ mutually local monopoles are massless. We can represent massless monopole constraint as,

$$\begin{aligned} y^2 &= B_{2N+2}^2(x) - 4\Lambda^{4N+4} = \left(x^2 P_{2N}(x) + 2\Lambda^{2N+2}\right)^2 - 4\Lambda^{4N+4} \\ &= x^2 H_{2N-2n}^2(x) F_{2(2n+1)}(x) \end{aligned} \tag{3.1}$$

where we define $B_{2N+2}(x) \equiv x^2 P_{2N}(x) + 2\Lambda^{2N+2}$ and

$$H_{2N-2n}(x) = \prod_{i=1}^{N-n} (x^2 - p_i^2), \qquad F_{2(2n+1)}(x) = \prod_{i=1}^{2n+1} (x^2 - q_i^2).$$

The reason for introducing a polynomial $B_{2N+2}(x)$ rather than $P_{2N}(x)$ is that one can treat a curve (3.1) in a symmetric way and one can factorize it. The function $H_{2N-2n}(x)$ is a polynomial in $x$ of degree $(2N-2n)$ giving $(2N-2n)$ double roots and the function $F_{2(2n+1)}(x)$ is a polynomial in $x$ of degree $(4n+2)$. These are even functions in $x$ and depend on only $x^2$. The characteristic function $P_{2N}(x)$ becomes

$$P_{2N}(x) = \det(x - \Phi) = \prod_{I=1}^{N} (x^2 - \phi_I^2).$$

When the degree $(2k+1)$ of $W'(x)$ is equal to $(2n+1)$, the highest $(2n+2)$ coefficients of $F_{2(2n+1)}$ are given in terms of $W'(x)$ [11] and the expression (2.4) is the same.

• **Superpotential of degree $2(k+1)$ less than $2N$**

We describe the superpotential when the degree of $W'$, $(2k+1)$ is greater than $(2n+1)$ with tree level superpotential (2.1), namely $2n+2 \leq 2k+2 \leq 2N$ case. Let us consider the superpotential under these constraints,

$$W_{eff} = \sum_{r=1}^{k+1} g_{2r} u_{2r} + \sum_{i=0}^{2N-2n} \left[ L_i \oint \frac{B_{2N+2}(x) - 2\epsilon_i \Lambda^{2N+2}}{(x-p_i)} dx + B_i \oint \frac{B_{2N+2}(x) - 2\epsilon_i \Lambda^{2N+2}}{(x-p_i)^2} dx \right],$$

where $L_i$ and $B_i$ are Lagrange multipliers and $\epsilon_i = \pm 1$ as we have considered in $SO(2N)$ case. The convention of an assignment of $\epsilon_i$ is different from the one in [11]. The $p_i$'s are the locations of the double roots of $y^2 = B_{2N+2}^2(x) - 4\Lambda^{4N+4}$. The massless monopoles occur in pair $(p_i, -p_i)$ and also both the expression $B_{2N+2}(x) - 2\epsilon_i \Lambda^{2N+2}$ at $x = \pm p_i$ and its derivative with respect to

$x$ at $x = \pm p_i$ are vanishing since $B_{2N+2}(x)$ is an even function of $x$. The half of the Lagrange multipliers are not independent. Totally the number of constraint is $(N-n)$.

The variation of $W_{eff}$ with respect to $B_i$ leads to

$$\begin{aligned} 0 &= \oint \frac{B_{2N+2}(x) - 2\epsilon_i \Lambda^{2N+2}}{(x-p_i)^2} dx = B'_{2N+2}(x)|_{x=p_i} = \Big(x^2 P'_{2N}(x) + 2x P_{2N}(x)\Big)|_{x=p_i} \\ &= x^2 P_{2N}(x) \left( \mathrm{Tr}\frac{1}{x-\Phi} + \frac{2}{x} \right)|_{x=p_i} \end{aligned} \tag{3.2}$$

where we exploited the equation of motion for $L_i$ and the last equality comes from the following relation,

$$\mathrm{Tr}\frac{1}{x-\Phi} = \sum_{I=1}^{N} \frac{2x}{x^2 - \phi_I^2}.$$

Since $P_{2N}(x = p_i) \neq 0$ due to the relation (3.1) and $H_{2N-2n}(x = p_i) = 0$, we arrive at

$$\left( \mathrm{Tr}\frac{1}{x-\Phi} + \frac{2}{x} \right)|_{x=p_i} = 0.$$

Notice that the presence of $2/x$ with positive sign is different from the one $-2/x$ for $SO(2N)$ case and $-1/x$ for $SO(2N+1)$ case. All the property of the characteristic function $P_{2N}(x)$ for the previous cases hold for $Sp(2N)$ case also. We do not write them explicitly here. Next we consider the variations of $W_{eff}$ with respect to $p_j$,

$$\begin{aligned} 0 &= L_j \oint \frac{B_{2N+2}(x) - 2\epsilon_i \Lambda^{2N+2}}{(x-p_i)^2} dx + 2B_j \oint \frac{B_{2N+2}(x) - 2\epsilon_i \Lambda^{2N+2}}{(x-p_i)^3} dx \\ &= 2B_j \oint \frac{B_{2N+2}(x) - 2\epsilon_i \Lambda^{2N+2}}{(x-p_i)^3} dx \end{aligned}$$

where in the last equality we used the equation of motion for $B_i$ (3.2). In general, this integral does not vanish, then we have $B_i = 0$ based on the same reason about the structure of the algebraic curve in previous cases. Let us consider variations with respect to $u_{2r}$,

$$0 = g_{2r} - \sum_{i=0}^{2N-2n} \oint \left[ \frac{x^2 P_{2N}(x)}{x^{2r}} \right]_+ \frac{L_i}{x-p_i} dx,$$

where we used $B_i = 0$ at the level of equation of motion and we express the explicit form for $B_{2N+2}(x)$ in terms of $P_{2N}(x)$. Multiplying this with $z^{2r-1}$ and summing over $r$, we can obtain the following relation,

$$W'(z) = \sum_{r=1}^{k+1} g_{2r} z^{2r-1} = \sum_{i=0}^{2N-2n} \oint \sum_{r=1}^{k+1} z^{2r-1} \frac{x^2 P_{2N}(x)}{x^{2r}} \frac{L_i}{(x-p_i)} dx. \tag{3.3}$$

Let us introduce a new polynomial $Q(x)$ defined as

$$\sum_{i=0}^{2N-2n} \frac{x L_i}{(x-p_i)} = L_0 + \sum_{i=1}^{N-n} x L_i \left( \frac{1}{x-p_i} + \frac{1}{x+p_i} \right) = L_0 + \sum_{i=1}^{N-n} \frac{2x^2 L_i}{x^2 - p_i^2} \equiv \frac{Q(x)}{H_{2N-2n}(x)}. \tag{3.4}$$

Using this new function we can rewrite (3.3) as

$$W'(z) = \oint \sum_{r=1}^{k+1} \frac{z^{2r-1}}{x^{2r}} \frac{Q(x) x^2 P_{2N}(x)}{x H_{2N-2n}(x)} dx.$$

Since $W'(z)$ is a polynomial of degree $(2k+1)$, the order of $Q(x)$ is $(2k-2n)$, so we denote it by $Q_{2k-2n}$. Thus we have found the order of polynomial $Q(x)$ and the order of integrand in (3.3) behaves like as $\mathcal{O}(x^{2k-2r+3})$. Thus if $r \geq k+2$ it does not contribute to the integral since the Laurent expansion around the origin vanishes. We can replace the upper value of summation to infinity.

$$W'(z) = \oint \sum_{r=1}^{\infty} \frac{z^{2r-1}}{x^{2r}} \frac{Q_{2k-2n}(x) x^2 P_{2N}(x)}{x H_{2N-2n}(x)} dx = \oint z \frac{Q_{2k-2n}(x) x^2 P_{2N}(x)}{x(x^2 - z^2) H_{2N-2n}(x)} dx. \tag{3.5}$$

From (3.1) one can write,

$$x^2 P_{2N}(x) = x \sqrt{F_{2(2n+1)}(x)} H_{2N-2n}(x) + \mathcal{O}(x^{-2N-2}).$$

By substituting this relation to (3.5), the $\mathcal{O}(x^{-2N-2})$ terms do not contribute the integral, so we can drop those terms.

Therefore we have

$$W'(z) = \oint z \frac{y_m(x)}{x^2 - z^2} dx, \qquad y_m^2(x) = F_{2(2n+1)}(x) Q_{2k-2n}^2(x)$$

corresponding to the equation of motion and the curve in the matrix model context. Then we get the final result

$$y_m^2(x) = F_{2(2n+1)}(x) Q_{2k-2n}^2(x) = W'^2_{2k+1}(x) + \mathcal{O}(x^{2k}) = W'^2_{2k+1}(x) + f_{2k}(x)$$

where both $F_{2(2n+1)}(x)$ and $Q_{2k-2n}(x)$ are functions of $x^2$, then $f_{2k}(x)$ is also a function of $x^2$. When $2k = 2n$, we reproduce (2.4) with $Q_0 = g_{2n+2}$. The above relation determines a polynomial $F_{2(2n+1)}(x)$ in terms of $(2n+1)$ unknown parameters by assuming the leading coefficient to be normalized by 1 by assuming that $W(x)$ is known.

• **Superpotential of degree $2(k+1)$ where $k$ is arbitrary large**

Let us extend the study of previous discussion to the general case without imposing the constraint on the degree of superpotential. As before, we denote $\frac{1}{2k}\langle \mathrm{Tr}\Phi^{2k} \rangle = U_{2k}$. The corresponding quantum expression can be written as

$$\left\langle \mathrm{Tr} \frac{1}{x - \Phi} \right\rangle = \frac{2N}{x} + \sum_{k=1}^{\infty} \frac{2k U_{2k}}{x^{2k+1}}$$

and also quantum mechanically this can be given by

$$\left\langle \mathrm{Tr} \frac{1}{x - \Phi} \right\rangle = \frac{d}{dx} \left[ \log \left( B_{2N+2}(x) + \sqrt{B_{2N+2}^2(x) - 4\Lambda^{4N+4}} \right) - \log x^2 \right]. \tag{3.6}$$

Note in order to make the degree of the expression inside of the log as $2N$, we inserted the last term. By integrating and exponentiating, one gets

$$x^{2N+2} \exp\left(-\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) = B_{2N+2}(x) + \sqrt{B_{2N+2}^2(x) - 4\Lambda^{4N+4}} \tag{3.7}$$

where $C$ is an integration constant determined by the semiclassical limit $\Lambda \to 0$:

$$Cx^{2N} \exp\left(-\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) = 2P_{2N}(x) \longrightarrow C = 2.$$

Solving (3.7) with respect to $P_{2N}(x)$, we get

$$P_{2N}(x) = x^{2N} \exp\left(-\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) + \frac{\Lambda^{4N+4}}{x^{2N+4}} \exp\left(\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right). \tag{3.8}$$

This looks similar to the previous expression. The power behavior of $\Lambda$ and $x$ in the second term is different. Since $P_{2N}(x)$ is a polynomial in $x$, the relation (3.8) can be used to express $U_{2r}$ with $2r > 2N$ in terms of $U_{2r}$ with $2r \leq 2N$ by imposing the vanishing of the negative powers of $x$.

Let us introduce a new polynomial whose coefficients are Lagrange multipliers. The superpotential with these constraints is described as

$$\begin{aligned} W_{eff} &= \sum_{r=1}^{k+1} g_{2r} u_{2r} + \sum_{i=0}^{2N-2n} \left[ L_i \oint \frac{B_{2N+2}(x) - 2\epsilon_i \Lambda^{2N+2}}{(x-p_i)} dx + B_i \oint \frac{B(x) - 2\epsilon_i \Lambda^{2N+2}}{(x-p_i)^2} dx \right] \\ &+ \oint R_{2k-2N+2}(x) \left[ x^{2N} \exp\left(-\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) + \frac{\Lambda^{4N+4}}{x^{2N+4}} \exp\left(\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right)\right] dx. \end{aligned}$$

Here $R_{2k-2N+2}(x)$ is a polynomial of degree $(2k - 2N + 2)$ whose coefficients are regarded as Lagrange multipliers which impose constraints $U_{2r}$ with $2r > 2N$ in terms of $U_{2r}$ with $2r \leq 2N$. The derivative of $W_{eff}$ with respect to $U_{2r}$ leads to

$$\begin{aligned} 0 &= g_{2r} + \oint \frac{R_{2k-2N+2}(x)}{x^{2r}} \left( -x^{2N} \exp\left(-\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) + \frac{\Lambda^{4N+4}}{x^{2N+4}} \exp\left(\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right)\right) dx \\ &+ \oint \sum_{i=0}^{2N-2n} \frac{x^2 L_i}{(x-p_i)} \frac{\partial P_{2N}(x)}{\partial U_{2r}} dx. \end{aligned} \tag{3.9}$$

Using (3.7) we have the relation,

$$-x^{2N+2} \exp\left(-\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) + \frac{x^2 \Lambda^{4N+4}}{x^{2N+4}} \exp\left(\sum_{i=1}^{\infty} \frac{U_{2i}}{x^{2i}}\right) = -\sqrt{B_{2N+2}^2(x) - 4\Lambda^{4N+4}}$$

and

$$\frac{\partial P_{2N}(x)}{\partial U_{2r}} = -\frac{P_{2N}(x)}{x^{2r}} \ \text{ for } \ 2r \leq 2N, \qquad \frac{\partial P_{2N}(x)}{\partial U_{2r}} = 0 \ \text{ for } \ 2r > 2N.$$

Using these relations we can rewrite (3.9) as follows:

$$0 = g_{2r} + \oint \frac{R_{2k-2N+2}(x)}{x^{2r}} \left(-\frac{1}{x^2}\sqrt{B_{2N+2}^2(x) - 4\Lambda^{4N+4}}\right) dx - \oint \sum_{i=0}^{2N-2n} \frac{x^2 L_i}{x-p_i} \frac{P_{2N}(x)}{x^{2r}} dx. \quad (3.10)$$

From the massless monopole constraint (3.1) we have the relation,

$$x^2 P_{2N}(x) = x H_{2N-2n}(x)\sqrt{F_{2(2n+1)}(x)} + \mathcal{O}(x^{-2N-2}). \quad (3.11)$$

Substituting (3.11) and (3.1) into (3.10), the $\mathcal{O}(x^{-2N-2})$ terms do not contribute to integral then we get an expression for the coupling $g_{2r}$

$$\begin{aligned} g_{2r} &= \oint \frac{R_{2k-2N+2}(x)}{x^{2r}} x H_{2N-2n}(x)\sqrt{F_{2(2n+1)}(x)} dx \\ &+ \oint \sum_{i=0}^{2N-2n} \frac{L_i}{(x-p_i)} \frac{x H_{2N-2n}(x)\sqrt{F_{2(2n+1)}(x)}}{x^{2r}} dx. \end{aligned}$$

As in previous analysis, we define a new polynomial $Q(x)$ in (3.4) and we see

$$\begin{aligned} W'(z) &= \sum_{r=1}^{k+1} \left[ \oint \frac{R_{2k-2N+2}(x)}{x^{2r}} x H_{2N-2n}(x)\sqrt{F_{2(2n+1)}(x)} dx \right. \\ &+ \left. \oint \sum_{i=0}^{2N-2n} \frac{L_i}{(x-p_i)} \frac{x H_{2N-2n}(x)\sqrt{F_{2(2n+1)}(x)}}{x^{2r}} dx \right] z^{2r-1} \\ &= \oint \frac{z}{x^2 - z^2} \sqrt{F_{2(2n+1)}(x)} \left[ R_{2k-2N+2}(x) H_{2N-2n}(x) + Q_{2N-2n}(x) \right] dx \end{aligned}$$

if the matrix model curve is given by

$$\begin{aligned} y_m^2 &= F_{2(2n+1)}(x) \widetilde{Q}_{2k-2n+2}^2(x) \\ &\equiv F_{2(2n+1)}(x) \left( R_{2k-2N+2}(x) H_{2N-2n}(x) + Q_{2N-2n}(x) \right)^2. \end{aligned}$$

When $2n = 2N$ (no massless monopoles), $\widetilde{Q}_{2k-2N+2}(x) = R_{2k-2N+2}(x)$. When the degree of superpotential is equal to $2N$, in other words, $2k = 2N-2$, then $\widetilde{Q}_{2N-2n}(x) = R_0 H_{2N-2n}(x) + Q_{2N-2n}(x)$. In particular, for $2n = 2N$,

$$y_m^2(x) = F_{4N+2}(x) = x^{-2} \left( B_{2N+2}^2(x) - 4\Lambda^{4N+4} \right).$$

- **A generalized Konishi anomaly**

Now we are ready to study the generalized Konishi anomaly equation. As in [26], we restrict to the case with $\langle \mathrm{Tr} W'(\Phi) \rangle = \mathrm{Tr} W'(\Phi_{cl})$ and assume that the degree of superpotential is less than $2N$. One can write

$$W'(\phi_I) = \sum_{i=0}^{2N-2n} \oint \phi_I \frac{x^2 P_{2N}(x)}{(x^2 - {\phi_I}^2)} \frac{L_i}{(x-p_i)} dx. \quad (3.12)$$

Using this equation we have following relations,

$$
\begin{aligned}
\mathrm{Tr}\frac{W'(\Phi_{cl})}{x-\Phi_{cl}} &= \mathrm{Tr}\sum_{k=0}^{\infty} z^{-k-1}\Phi_{cl}^{k} W'(\Phi_{cl}) = \sum_{k=0}^{\infty} z^{-(2i+1)-1} 2\sum_{I=1}^{N}\phi_I^{2i+1} W'(\phi_I) \\
&= 2\sum_{I=1}^{N}\phi_I W'(\phi_I)\frac{1}{(z^2-\phi_I^2)} \\
&= \sum_{I=1}^{N}\frac{2\phi_I^2}{(z^2-\phi_I^2)}\sum_{i=0}^{2N-2n}\oint \frac{x^2 P_{2N}(x)}{(x^2-\phi_I^2)}\frac{L_i}{(x-p_i)}dx,
\end{aligned}
\tag{3.13}
$$

where $z$ is outside the contour of integration. Using (2.7) we can write this factor as

$$
\frac{2\phi_I^2}{(z^2-\phi_I^2)(x^2-\phi_I^2)} = \frac{1}{(x^2-z^2)}\left(z\mathrm{Tr}\frac{1}{z-\Phi} - x\mathrm{Tr}\frac{1}{x-\Phi}\right).
$$

Thus we can write (3.13) as

$$
\mathrm{Tr}\frac{W'(\Phi_{cl})}{x-\Phi_{cl}} = \oint \sum_{i=0}^{2N-2n}\frac{x^2 P_{2N} L_i}{(x^2-z^2)(x-p_i)}\left(z\mathrm{Tr}\frac{1}{z-\Phi} - x\mathrm{Tr}\frac{1}{x-\Phi}\right)dx. \tag{3.14}
$$

As in the case [26] we can rewrite the contour of the integral,

$$
\oint_{z_{out}} = \oint_{z_{in}} - \oint_{C_z+C_{-z}}
$$

where $C_z$ and $C_{-z}$ are small contour around $z$ and $-z$ respectively. Thus the first term in (3.14) can be rewritten as

$$
\mathrm{Tr}\frac{1}{z-\Phi}\oint_{z_{out}}\frac{zQ_{2k-2n+2}(x)x^2P_{2N}(x)}{H_{2N-2n}(x)x(x^2-z^2)}dx.
$$

From the change of an integration, this is given by

$$
\begin{aligned}
&\mathrm{Tr}\frac{1}{z-\Phi}\left(\oint_{z_{in}}\frac{zQ_{2k-2n+2}(x)x^2P_{2N}(x)}{H_{2N-2n}(x)x(x^2-z^2)}dx + \oint_{C_z+C-z}\frac{zQ_{2k-2n+2}(x)x^2P_{2N}(x)}{H_{2N-2n}(x)x(x^2-z^2)}dx\right) \\
&= \mathrm{Tr}\frac{1}{z-\Phi}\left(W'(z) - \frac{y_m(z)z^2P_{2N}(z)}{\sqrt{B_{2N+2}^2(z)-4\Lambda^{4N+4}}}\right),
\end{aligned}
$$

in the last equality we used (3.12) and

$$
H_{2N-2n}(z) = \frac{\sqrt{B_{2N+2}^2(z)-4\Lambda^{4N+4}}}{z\sqrt{F_{2(2n+1)}(z)}}, \qquad y_m^2(z) = F_{2(2n+1)}(z)Q_{2k-2n+2}^2(z).
$$

We now make an integration over $x$

$$
-\sum_{i=0}^{2N-2n}\oint\frac{L_i x^2 P_{2N}(x)}{(x-p_i)x(x^2-z^2)}\mathrm{Tr}\frac{x}{x-\Phi}dx = -\sum_{i=0}^{2N-2n}\frac{L_i p_i^2 P_{2N}(x=p_i)}{p_i^2-z^2}\mathrm{Tr}\frac{p_i}{p_i-\Phi}.
$$

By using the equation of motion for $B_i$ to change the trace part, one gets

$$\sum_{i=0}^{2N-2n} 2\frac{L_i p_i^2 P_{2N}(x=p_i)}{p_i^2 - z^2} = + \sum_{i=0}^{2N-2n} \oint \frac{2x^2 P_{2N}(x)}{x^2 - z^2}\frac{L_i}{x-p_i}dx$$

where note that $z$ is *outside* the contour of integration in last equation. Therefore as in $SO$ case we can rewrite the last equation as follows:

$$+2\frac{W'(z)}{z} - \sum_{i=0}^{2N-2n} \oint_{C_z + C_{-z}} \frac{2x^2 P_{2N}(x)}{(x^2 - z^2)}\frac{L_i}{(x-p_i)}dx = +2\frac{W'(z)}{z} - \frac{2}{z}\frac{y_m z^2 P_{2N}(z)}{\sqrt{B_{2N+2}^2(z) - 4\Lambda^{4N+4}}}$$

Therefore we obtain

$$\mathrm{Tr}\frac{W'(\Phi_{cl})}{z - \Phi_{cl}} = \mathrm{Tr}\frac{1}{z-\Phi_{cl}}\left(W'(z) + \frac{y_m(z) z^2 P_{2N}(z)}{\sqrt{B_{2N+2}^2(z) - 4\Lambda^{4N+4}}}\right) + 2\frac{W'(z)}{z} - \frac{2}{z}\frac{y_m z^2 P_{2N}(z)}{\sqrt{B_{2N+2}^2(z) - 4\Lambda^{4N+4}}}.$$

Remembering that

$$\mathrm{Tr}\frac{1}{z-\Phi_{cl}} = \frac{P'_{2N}(z)}{P_{2N}(z)},$$

the second and fourth terms are rewritten as follows;

$$\mathrm{Tr}\frac{1}{z-\Phi_{cl}}\frac{y_m(z) z^2 P_{2N}(z)}{\sqrt{B_{2N+2}^2(z) - 4\Lambda^{4N+4}}} - \frac{2}{z}\frac{y_m z^2 P_{2N}(z)}{\sqrt{B_{2N}^2(z) - 4\Lambda^{4N+4}}} = -\frac{2y_m}{z} - \left\langle \mathrm{Tr}\frac{y_m}{z-\Phi}\right\rangle$$

where we used (3.6).

Taking into account the relation,

$$\mathrm{Tr}\frac{W'(\Phi_{cl}) - W'(z)}{z-\Phi_{cl}} = \left\langle \mathrm{Tr}\frac{W'(\Phi) - W'(z)}{z-\Phi}\right\rangle$$

one can summarize as follows,

$$\left\langle \mathrm{Tr}\frac{W'(\Phi)}{z-\Phi}\right\rangle \;=\; \left(\left\langle \mathrm{Tr}\frac{1}{z-\Phi}\right\rangle + \frac{2}{z}\right)[W'(z) - y_m(z)]$$

This is the generalized Konishi anomaly equation for $Sp(2N)$ case.

## 3.2 The multiplication map and the confinement index

Although in this subsection, we discuss $Sp(2N)$ gauge theories, the matrix model curve which derived in 3.1 is the same as $SO(2N)$ case (2.27). Thus we can use the same notations and ideas introduced in $SO(N)$ case. The symmetry breaking pattern is given by

$$Sp(2N) \to Sp(2N_0) \times \prod_{i=1}^{n} U(N_i). \qquad (3.15)$$

The difference between $SO(2N)$ case and $Sp(2N)$ case is the contribution of unoriented diagram to the effective superpotential,

$$W_{eff}(S_i) = (N_0+2)\frac{\partial \mathcal{F}}{\partial S_i} + \sum_{i=1}^{n} N_i \frac{\partial \mathcal{F}}{\partial S_i} + 2\pi i \tau_0 \sum_{i=0}^{n} S_i + 2\pi i \sum_{i=1}^{n} b_i S_i. \tag{3.16}$$

Thus if we introduce $\hat{N}_0 \equiv 2N_0 + 2$ and $\hat{N}_i = N_i$, we can study the phase structure in the same way as $SO(2N)$ case. The criterion of confinement is that if three objects $(2N_0+2)$, $N_i$ and $b_i$ have common divisor, the vacua are confining vacua, if not, it is called Coulomb vacua.

For $Sp(2N)$ case we can construct multiplication map in the same way as $SO(2N)$ and $SO(2N+1)$ cases. By assumption we have the characteristic function $P_{2N}(x)$ that satisfies the following massless monopole constraint,

$$\left(x^2 P_{2N}(x) + 2\Lambda_0^{2N+2}\right)^2 - 4\Lambda_0^{4N+4} = x^2 H_{2N-2n}^2(x) F_{2(2n+1)}(x).$$

By using Chebyshev polynomial, let us consider a solution of $Sp(2KN+2K-2)$ gauge theory as

$$P_{2KN+2K-2}(x) = \frac{2\eta^{2K}\Lambda^{2KN+2K}}{x^2}\mathcal{T}_K\left(\frac{x^2 P_{2N}(x)}{2\eta^2\Lambda^{2N+2}} + 1\right) - \frac{2\Lambda^{2KN+2K}}{x^2} \tag{3.17}$$

where we introduce

$$\widetilde{x} = \frac{x^2 P_{2N}(x)}{2\eta^2\Lambda^{2N+2}} + 1$$

and $\eta^{2K} = 1$. One can check that in the right hand side of (3.17), the argument $\widetilde{x}$ of the first kind Chebyshev polynomial has a degree $(2N+2)$, by the definition of the first kind Chebyshev polynomial and the right hand side has a factor $1/x^2$, therefore it leads to a polynomial of degree $-2+K(2N+2)$ totally. It is right to write the left hand side as $P_{2KN-2K+2}(x)$ which agrees with the number of order in $x$ in the right hand side. The power of $\Lambda$ for the $\widetilde{x}$ of Chebychev polynomial was fixed by the power of $x$ which is equal to $(2N+2)$. The power of $\Lambda$ in front of the Chebyshev polynomial can be fixed by the dimension consideration of both sides. The right hand side should contain $(2KN+2K-2)$ which is equal to $-2+(2KN+2K)$ where the first term comes from the power of $x$ and the second one should be the power of $\Lambda$. Also note that the same $\eta$ term appears in the denominator of the argument of Chebyshev polynomial. Contrary to the previous $SO(2N)$ or $SO(2N+1)$ cases, the curve contains $\Lambda^{4N+4}$ as well as $\Lambda^{2N+2}$. In order to get the correct number of vacua of $Sp(2KN+2K-2)$ gauge theory, it is very important to know the right behavior of $\eta$ and $\Lambda$ in $\widetilde{x}$. The last term in (3.17) is included since the monopole constraint for $Sp(2N)$ case has an extra term in the left hand side, contrary to the $SO(2N)$ or $SO(2N+1)$ cases. Let us check that (3.17) is indeed a solution of the massless monopole constraint of $Sp(2KN+2K-2)$ gauge theory. By introducing a new function $B_{2KN+2K}(x) = x^2 P_{2KN+2K-2}(x) + 2\Lambda^{2KN+2K}$, we can see the following relations from the solution (3.17). Then one can construct

$$\begin{aligned}
B^2_{2KN+2K}(x) - 4\Lambda^{4KN+4K} &= 4\Lambda^{4KN+4K}\left(\mathcal{T}_K^2(\widetilde{x}) - 1\right) \\
&= 4\Lambda^{4KN+4K}\left(\widetilde{x}^2 - 1\right)\mathcal{U}^2_{K-1}(\widetilde{x}) \\
&= \frac{\Lambda^{4KN+4K}}{\eta^4\Lambda^{4N+4}}\left[\left(x^2 P_{2N}(x) + 2\eta^2\Lambda^{2N+2}\right)^2 - 4\eta^4\Lambda^{4N+4}\right]\mathcal{U}^2_{K-1}(\widetilde{x}) \\
&= x^2\left[\eta^{-2}\Lambda^{2(K-1)(N+1)} H_{2N-2n}(x)\mathcal{U}_{K-1}(\widetilde{x})\right]^2 F_{2(2n+1)}(x).
\end{aligned}$$

In the fourth equality we used the identification $\Lambda_0^{2N+2} = \eta^2\Lambda^{2N+2}$. In other words, the relations

$$\begin{aligned}
P_{2KN+2K-2}(x) &= \frac{2\eta^{2K}\Lambda^{2KN+2K}}{x^2}\mathcal{T}_K\left(\frac{x^2 P_{2N}(x)}{2\eta^2\Lambda^{2N+2}} + 1\right) - \frac{2\Lambda^{2KN+2K}}{x^2}, \\
\widetilde{F}_{2(2n+1)}(x) &= F_{2(2n+1)}(x), \\
H_{K(2N-2)-2n}(x) &= \eta^{-2}\Lambda^{2(K-1)(N+1)} H_{2N-2n}(x)\mathcal{U}_{K-1}\left(\frac{x^2 P_{2N}(x)}{2\eta^2\Lambda^{2N+2}} + 1\right)
\end{aligned}$$

satisfy

$$\left(x^2 P_{2KN+2K-2}(x) + 2\Lambda^{2NK+2K}\right)^2 - 4\Lambda^{4NK+4K} = x^2 H^2_{K(2N-2)-2n}(x)\widetilde{F}_{2(2n+2)}(x).$$

As in $SO(2N)$ or $SO(2N+1)$ case, since $\widetilde{F}_{2(2n+1)}(x) = F_{2(2n+1)}(x)$, the vacua constructed this way for the $Sp(2KN+2K-2)$ theory have the *same* superpotential as the vacua of the $Sp(2N)$ theory.

Next we consider the multiplication map of $T(x)$ ,

$$\begin{aligned}
T(x) &= \frac{d}{dx}\log\left[\left(B_{2N+2}(x) + \sqrt{B^2_{2N+2}(x) - 4\Lambda^{4N+4}}\right) - \log x^2\right] \\
&= \frac{B'_{2N+2}(x)}{\sqrt{B^2_{2N+2}(x) - 4\Lambda^{4N+4}}} - \frac{2}{x}.
\end{aligned}$$

The Seiberg-Witten differential is given by

$$d\lambda_{SW} = \frac{x dx B'_{2N+2}(x)}{\sqrt{B^2_{2N+2}(x) - 4\Lambda^{4N+4}}}.$$

By using the equation (3.17) we have a function $B_{2KN+2K}(x)$ after the map, which denoted as $B_{2KN+2K}(x) \equiv (x^2 P_{2KN+2K-2} + 2\Lambda^{2KN+2K})$. Let us consider

$$B_{2KN+2K}(x) = 2\eta^{2K}\Lambda^{2KN+2K}\mathcal{T}_K\left(\frac{x^2 P_{2N}(x)}{2\eta^2\Lambda^{2N+2}} + 1\right).$$

The derivative of $B_{2KN+2K}(x)$ with respect to $x$ leads to

$$B'_{2KN+2K}(x) = \eta^{2K}\Lambda^{2KN+2K} K\mathcal{U}_{K-1}(\widetilde{x})\left(\frac{x^2 P_{2N}(x)}{\eta^2\Lambda^{2N+2}}\right)'.$$

Also we have the relation

$$\sqrt{B_{2KN+2K}^2(x) - 4\Lambda^{4N+4}} = \frac{\Lambda^{2KN+2K}}{\eta^{2+2K}\Lambda^{2N+2}}\mathcal{U}_{K-1}(\widetilde{x})\sqrt{B_{2N+2}^2(x) - 4\Lambda_0^{4N+4}}.$$

Combining these two relations we have

$$\frac{B'_{2KN+2K}(x)}{\sqrt{B_{2KN+2K}^2(x) - 4\Lambda^{4KN+4K}}} = K\frac{B'_{2N+2}(x)}{\sqrt{B_{2N+2}^2(x) - 4\Lambda^{4N+4}}}.$$

Thus finally we obtain multiplication map of $T(x)$,

$$T_K(x) = KT(x) + K\frac{2}{x} - \frac{2}{x} \iff 2N'_0 + 2 = K(2N_0+2),\ N'_i = KN_i.$$

This means that under the multiplication map the special combination $(2N_0+2)$ have simple multiplication by $K$. As in $SO(2N)$ case the number of $Sp(2KN+2K-2)$ vacua with confinement index $K$ is $K$ times the one of $Sp(2N)$ Coulomb vacua. All of these confining vacua can be constructed by this map. Let us denote adjoint chiral superfield as $\Phi_0$ in the $Sp(2N)$ gauge theory. Then through the multiplication map we constructed the following quantum operator in the $Sp(2KN+2K-2)$ gauge theory can be obtained by simply multiplying the confinement index $K$ by the corresponding operator in the $Sp(2N)$ gauge theory as follows

$$\left\langle \mathrm{Tr}\frac{1}{x-\Phi}\right\rangle + \frac{2}{x} = K\left(\left\langle \mathrm{Tr}\frac{1}{x-\Phi_0}\right\rangle + \frac{2}{x}\right).$$

### 3.3 Examples

In this subsection we will analyze some examples of $Sp(2N)$ gauge theory with rank $n=1$, namely $Sp(2N)$ gauge group is broken to $Sp(2N_0)\times U(N_1)$. We will deal with $\mathcal{N}=1$ gauge theories that are $\mathcal{N}=2$ theories deformed by tree level superpotential characterized by

$$W(\Phi) = \frac{m}{2}\mathrm{Tr}\Phi^2 + \frac{g}{4}\mathrm{Tr}\Phi^4.$$

For simplicity we consider the special case $k=n=1$, namely the number of critical point of $W(x)$ equal to the one of gauge groups after Higgs breaking.

As already discussed, massless monopole constraint with $n=1$ is given by

$$B_{2N+2}^2(x) - 4\Lambda^{4N+4} = \left[xH_{2N-2}(x)\right]^2 F_6(x). \tag{3.18}$$

Equivalently we can factorize this equation as,

$$B_{2N+2}(x) + 2\Lambda^{2N+2} = H_{s_+}^2(x)R_{2N+2-2s_+}(x), \tag{3.19}$$

$$B_{2N+2}(x) - 2\Lambda^{2N+2} = H_{s_-}^2(x)\widetilde{R}_{2N+2-2s_-}(x), \tag{3.20}$$

$$xH_{2N-2}(x) = H_{s_+}(x)H_{s_-}(x), \qquad F_6(x) = R_{2N+2-2s_+}(x)\widetilde{R}_{2N+2-2s_-}(x).$$

We want to point out the relation to $SO(2N)$ case. Comparing (3.19)and (3.20) with (2.39) and (2.40), we find the similarities and correspondences in $P_{2N-2}(x) \iff B_{2N+2}(x)$. So massless monopole constraint for $SO(2N+2)$ case is the same as the one for $Sp(2N-2)$ case. However the subtle difference comes from the function $B_{2N+2}(x) = x^2 P_{2N}(x) + 2\Lambda^{2N+2}$. The left hand side in (3.20) leads to

$$B_{2N+2}(x) - 2\Lambda^{2N+2} = x^2 P_{2N}(x).$$

Thus we must have a factor $x^2$ in the right hand side of (3.20). Thus we cannot describe two case $(s_+, s_-) = (a, b)$ and $(b, a)$ in the same way by using $\eta = \pm 1$. Taking into account this constraint, we can rewrite massless monopole constraint (3.19) and (3.20) as

$$\begin{aligned} B_{2N+2}(x) + 2\Lambda^{2N+2} &= H^2_{2s_+}(x) R_{2N+2-4s_+}(x), \\ B_{2N+2}(x) - 2\Lambda^{2N+2} &= x^2 H^2_{2s_-}(x) \widetilde{R}_{2N-4s_-}(x), \\ H_{2N-2}(x) = H_{2s_+}(x) H_{2s_-}(x), \qquad & F_6(x) = R_{2N+2-4s_+}(x) \widetilde{R}_{2N-4s_-}(x) \end{aligned} \tag{3.21}$$

where $2s_+ + 2s_- = 2N - 2$ and $2N - 4s_- \geq 0$ and $2N + 2 - 4s_+ \geq 0$. From the first two equations (3.21) we have a constraint that is useful for analysis below,

$$H^2_{2s_+}(x) R_{2N+2-4s_+}(x) - 4\Lambda^{2N+2} = x^2 H^2_{2s_-}(x) \widetilde{R}_{2N-4s_-}(x). \tag{3.22}$$

After solving this equation, we can find matrix model curve,

$$y^2_m(x) = R_{2N+2-4s_+}(x) \widetilde{R}_{2N-4s_-}(x),$$

and

$$R_{2N+2-4s_+}(x) \widetilde{R}_{2N-4s_-}(x) = W_3'^2(x) + f_2(x). \tag{3.23}$$

As in $SO(2N)$ or $SO(2N+1)$ case, from the coefficient of $x^2$ in $f_2(x)$ we can read off the expectation value of glueball superfield, $S = S_0 + S_1$. Let us start with the explicit analysis of $Sp(2)$, $Sp(4)$ and $Sp(6)$.

- $Sp(2)$ **case**

The first example is $Sp(2)$ gauge theory. In this case massless monopole constraint (3.18) is trivial since $2N - 2n = 2 - 2 = 0$. If we parameterize characteristic function as $P_2(x) = x^2 - v^2$, we can rewrite as, from (3.21),

$$\begin{aligned} B_4(x) + 2\Lambda^4 &= x^2(x^2 - v^2) + 4\Lambda^4 = R_4(x), \\ B_4(x) - 2\Lambda^4 &= x^2(x^2 - v^2) = x^2 \widetilde{R}_2(x). \end{aligned}$$

Thus the matrix model curve is given by

$$\begin{aligned} F_6(x) &= y_m^2 = x^2(x^2-v^2)^2 + 4\Lambda^2(x^2-v^2), \\ W_3'(x) &= x(x^2-v^2), \\ f_2(x) &= 4\Lambda^2(x^2-v^2), \\ S &= -\Lambda^2. \end{aligned}$$

Then there is only one vacuum for given $W(x)$. In the semiclassical limit $\Lambda \to 0$ the characteristic function becomes $P_2(x) \to (x^2-v^2)$. So in this vacuum, the gauge group $Sp(2)$ breaks into $U(1)$.

- $Sp(4)$ **case**

The second example is a $Sp(4)$ gauge theory where $2N-2n = 4-2 = 2$. As already discussed, this case corresponds to $SO(8)$ case and is somewhat interesting example. The number of massless monopoles is determined by $2s_+ + 2s_- = 2$. Thus we have two branches $(s_+, s_-) = (1,0)$ and $(0,1)$. At first, we study the case $(1,0)$. We can parameterize the massless monopole constraint as

$$\begin{aligned} B_6(x) + 2\Lambda^6 &= (x^2-a^2)^2(x^2+A), \\ B_6(x) - 2\Lambda^6 &= x^2\left(x^4 + Bx^2 + C\right). \end{aligned}$$

These parameters must satisfy (3.23). The solution for the constraint is given by, from (3.22),

$$\begin{aligned} B_6(x) + 2\Lambda^6 &= (x^2-a^2)^2\left(x^2 + \frac{4\Lambda^6}{a^4}\right), \\ B_6(x) - 2\Lambda^6 &= x^2\left[(x^2-a^2)^2 + \frac{4\Lambda^6}{a^4}(x^2-2a^2)\right]. \end{aligned}$$

We can find the matrix model curve from the solution as,

$$y_m^2(x) = \left(x^2 + \frac{4\Lambda^6}{a^4}\right)\left[(x^2-a^2)^2 + \frac{4\Lambda^6}{a^4}(x^2-2a^2)\right]$$

and from this one gets

$$\begin{aligned} W_3'(x) &= x\left(x^2 + \frac{4\Lambda^6}{a^4} - a^2\right), \\ f_2(x) &= -\frac{8\Lambda^6}{a^2}x^2 + \frac{4\Lambda^6}{a^2}\left(a^2 - 2\frac{4\Lambda^6}{a^4}\right), \\ S &= \frac{2\Lambda^6}{a^2}. \end{aligned}$$

For these vacua we consider the semiclassical limit. In this case there exist two limits.

1. Fix a: The characteristic function behaves $P_4 \to (x^2 - a^2)^2$ and the gauge group $Sp(4)$ breaks into $U(2)$.

2. Fix $v \equiv \frac{4\Lambda^6}{a^4}$: The characteristic function goes to $P_4 \to x^2(x^2+v)$ and the $Sp(4)$ breaks into $Sp(2) \times U(1)$.

Thus we can transit continuously by changing the parameters. At last, as in $SO(8)$ case we count the number of vacua for fixed tree level superpotential, $W_3'(x) = x(x^2 + \Delta)$. From the previous result we can represent $\Delta$ as

$$\Delta = \frac{4\Lambda^6}{a^4} - a^2.$$

We evaluate this equation under the two semiclassical limit 1 and 2 discussed above.

1. In this limit since $\Delta = -a^2$ we have only one function $f_2(x)$. Thus we have one vacua. As we will see below we have one vacua that have the same gauge group. Thus all the vacua with this gauge group are two vacua. On the other hand, gauge group becomes $U(2)$ under this limit. So the number of vacua is two. This equal to the one derived from above.

2. In this limit since $a^2 = \left(\frac{4\Lambda^6}{\Delta}\right)^{\frac{1}{2}}$, we have two functions $f_2(x)$. In other words we have two vacua for each potential. This number is equal to the one derived from gauge group $Sp(2) \times U(1)$, $(N_0+1) \times N_1 = (1+1) \cdot 1 = 2$, because dual Coxeter number of $Sp(2N_0)$ gauge theories are $(N_0+1)$.

Next let us consider $(s_+, s_-) = (0,1)$. We can easily solve the massless monopole constraint as follows:

$$\begin{aligned} B_6(x) + 2\Lambda^6 &= x^2(x^2-a^2)^2 + 4\Lambda^6, \\ B_6(x) - 2\Lambda^6 &= x^2(x^2-a^2)^2. \end{aligned}$$

From this solution we can find matrix model curve

$$\begin{aligned} F_6(x) &= y_m^2(x) = x^2(x^2-a^2)^2 + 4\Lambda^6, \\ W_3'(x) &= x(x^2-a^2), \\ f_2(x) &= 4\Lambda^6, \\ S &= 0. \end{aligned}$$

In the semiclassical limit it is easy to see $P_4(x) \to (x^2-a^2)^2$, which shows that the gauge group $Sp(4)$ breaks into $U(2)$. This branch is considerably different as the $SO(8)$ branch $(s_+, s_-) = (1,2)$, though $(2,1)$ is exactly same as the $Sp(6)$ branch. This agrees with the comment under (3.20).

- $Sp(6)$ **case**

The next example is a $Sp(6)$ gauge theory, which is the most interesting example. As already discussed, this case corresponds to $SO(10)$ case. In this case the number of massless monopoles is given by $2s_+ + 2s_- = 4$ and $2 \geq s_+, \frac{3}{2} \geq s_-$. There are two branches $(s_+, s_-) = (2,0)$ and $(1,1)$ in this theory.

**1. Confining branch**

At first, let us study $(2,0)$ branch, there exists a solution

$$\begin{aligned} B_8(x) + 2\Lambda^8 &= (x^2-a)^2(x^2-b)^2, \\ B_8(x) - 2\Lambda^8 &= x^2(x^6 + Ax^4 + Bx^2 + C). \end{aligned}$$

From (3.22), the coefficients $A, B, C, b$ can be represented in terms of $a$ and they are

$$\begin{aligned} A &= -2a - 4\frac{\eta\Lambda^4}{a}, \qquad B = a^2 + 8\eta\Lambda^4 + 4\frac{\Lambda^8}{a^2}, \\ C &= -4a\eta\Lambda^4 - \frac{8\Lambda^8}{a}, \qquad b = 2\frac{\eta\Lambda^4}{a}, \end{aligned}$$

where $\eta$ is 2-th root of unity. Thus the characteristic function $P_6(x)$ becomes

$$P_6(x) = \frac{1}{x^2}\left[(x^2-a)^2\left(x^2 - \frac{2\eta\Lambda^4}{a}\right)^2 - 4\Lambda^8\right]. \tag{3.24}$$

From these solutions we can find matrix model curve

$$\begin{aligned} y_m^2 &= x^2\left(x^2 + \frac{A}{2}\right)^2 + x^2\left(B - \frac{A^2}{4}\right) + C \\ &= x^2\left(x^2 - a - 2\frac{\eta\Lambda^4}{a}\right)^2 + 4x^2\eta\Lambda^4 - 4a\eta\Lambda^4 - \frac{8\Lambda^8}{a} \end{aligned}$$

and moreover we have

$$\begin{aligned} W_3'(x) &= x\left(x^2 - a - 2\frac{\eta\Lambda^4}{a}\right), \\ f_2(x) &= 4x^2\eta\Lambda^4 - 4a\eta\Lambda^4 - \frac{8\Lambda^8}{a}, \\ S &= -\eta\Lambda^4. \end{aligned}$$

Next we consider the semiclassical limit. There are two limits:

1. Fixed a: As $\Lambda \to 0$, the characteristic function has the relation $P_6(x) \to x^2(x^2-a)^2$. This means the gauge group $Sp(6)$ is broken to $Sp(2) \times U(2)$.

2. Fixed $w \equiv \frac{2\eta\Lambda^4}{a}$: The characteristic function goes to $P_6(x) \to x^2(x^2-w)^2$. This means the gauge group $Sp(6)$ is broken to $Sp(2) \times U(2)$. Thus we can see the continuous transition between two semiclassical limits. But both limits have the same gauge group. We want to survey whether these phases are the confining phase or not. Multiplication map for $Sp(2N)$

gauge theories has already been discussed, i.e. (3.17). If we choose $K=2$, we can construct the multiplication map from $Sp(2)$ to $Sp(6)$ where we denote as $P^{K=2}_{Sp(2)\to Sp(6)}(x)$,

$$\begin{aligned} P^{K=2}_{Sp(2)\to Sp(6)}(x) &= \frac{2\epsilon^4\Lambda^8}{x^2}\left[2\left(\frac{x^2P_2(x)}{2\epsilon^2\Lambda^4}+1\right)^2-1\right]-\frac{2\Lambda^8}{x^2} \\ &= \frac{1}{x^2}\left[(x^2(x^2-v^2)+2\epsilon^2\Lambda^4)^2-4\Lambda^8\right]. \end{aligned}$$

where $\epsilon$ is 4-th root of unity. If we choose $v^2=a+\frac{2\epsilon^2\Lambda^4}{a}$ and identify $\epsilon^2$ with $\eta$, we can reach the solution (3.24).

$$P^{K=2}_{Sp(2)\to Sp(6)}(x)=\frac{1}{x^2}\left[(x^2-a)^2\left(x^2-\frac{2\eta\Lambda^4}{a}\right)^2-4\Lambda^8\right].$$

Thus this vacua is a confining phase. After all we obtain continuous transition within the Coulomb phase that does not change the gauge group.

**2. Coulomb branch**

Next we consider the other branch $(s_+,s_-)=(1,1)$ and there are

$$\begin{aligned} B_8(x)+2\Lambda^8 &= (x^2-a)^2\left(x^4+Ax^2+C\right), \\ B_8(x)-2\Lambda^8 &= x^2(x^2-b)^2\left(x^2+D\right). \end{aligned}$$

As in previous examples, we must take into account the constraint (3.22). We get the equations,

$$a^2C=4\Lambda^8, \qquad a^2A-2aC-b^2D=0, \tag{3.25}$$

$$a^2-2aA-b^2+C+2bD=0, \qquad -2a+A+2b-D=0. \tag{3.26}$$

Although the solutions of these equations are a little bit complicated, those are reasonable solutions. We can continuously transit between the phases that can be obtained in semiclassical limit. So we concentrate on surveying the semiclassical limits of the solutions and do not obtain matrix model curves. From the first equation of (3.25) we can take two classical limits:

1. $\Lambda\to 0$ with $a\to 0$ and $C\to 0$,
2. $\Lambda\to 0$ with fixed $C$ and $a\to 0$.

To begin with we consider the case 1. In this case the equations (3.25) and (3.26) become

$$b=0, \qquad A=D.$$

Thus in this classical limit, the characteristic function behaves $P_6(x)\to x^4(x^2+A^2)$, which means that the gauge group breaks into $Sp(6)\to Sp(4)\times U(1)$.

Next we consider the case 2. The equations (3.25) and (3.26) become

$$b^2D=0, \qquad -b^2+C+2bD=0, \qquad A+2b-D=0.$$

Then we have the solutions,

$$D = 0, \qquad A = 2b, \qquad C = b^2.$$

So we obtain a relation $P_6(x) \to x^2(x^2-b)^2$, which means that $Sp(6) \to Sp(2) \times U(2)$. After all, we can transit continuously between the phases that have gauge groups $Sp(2) \times U(2)$ and $Sp(4) \times U(1)$.

So far we have studied the multiplication maps from $Sp(2)$ to $Sp(6)$ which is an example describing the vacua of $Sp(2KN+2K-2)$ gauge theory from $Sp(2N)$ theory. Then it is natural to consider whether there exists other multiplication map from $Sp(2N)$ to $Sp(2KN+2K-2)$ with different values of $N$ and $K$. For example, when $K=3$ and $N=1$, then the vacua of $Sp(10)$ gauge theory has the same superpotential as those of $Sp(2)$ theory. Therefore we expect that there exists a solution that is a confining phase from the explicit multiplication map from $Sp(2)$ to $Sp(10)$ by $K=3$ where we denote as $P^{K=3}_{Sp(2)\to Sp(10)}(x)$. Although we have considered a couple of examples, it would be interesting to study the possible confining vacua for each general $N$ and $K$ systematically.

We have seen that massless constraint for $SO(2N+2)$ is the same as the one for $Sp(2N-2)$. We ask whether there exists a multiplication map from $SO(2N)$ to $Sp(2M)$ or a map from $Sp(2N)$ to $SO(2M)$. From the experience we considered so far, let us consider the following characteristic function

$$P_{K(2N-2)-2}(x) = \frac{2\eta^{2K}\Lambda^{2KN-2K}}{x^2}\mathcal{T}_K\left(\frac{P_{2N}(x)}{2x^2\eta^2\Lambda^{2N-2}}\right) - \frac{2\Lambda^{2KN-2K}}{x^2}$$

where $P_{2N}(x)$ satisfies the usual massless monopole constraint of $SO(2N)$ gauge theory. From this relation one sees that a new function $P_{K(2N-2)-2}(x)$ satisfies the massless monopole constraint of $Sp(2KN-2K-2)$ gauge theory. For $Sp(6)$ case, namely $K(2N-2)-2=6$, if we choose $K=2$ we would expect to have a map from $SO(10) \to Sp(6)$. By using the explicit result for vacua of $SO(10)$ gauge theory that can be done by tedious calculations, we will see that $P^{K=2}_{SO(10)\to Sp(6)}(x)$ becomes the result for $Sp(6)$ which have already been derived. Conversely we can construct maps from $Sp(2N)$ to $SO(2M)$ in the same way as previous one. Of course the map with $K=1$ gives the same as the one that we have already obtained in terms of map from $SO(2N)$ to $Sp(2M)$ by substituting $K=1$. Although within same gauge groups the map with $K=1$ is trivial, the maps between the different gauge groups are somewhat interesting one, which maps the Coulomb vacua to the Coulomb vacua. This is naively expected because $K=1$ does not give a common divisor. So for the case $SO(2N)$ to $Sp(2M)$, we can expect to have the following relations, $2M+2 \iff K(2N-2)$. In other words, the effect of unoriented diagram or geometrically the effect of orientifold plane changes under these maps.

Finally, we would like to comment on the transition between different classical limit gauge groups $SO(N_0) \times U(N_1)$ and $SO(\widetilde{N}_0) \times U(\widetilde{N}_1)$ or $Sp(N_0) \times U(N_1)$ and $Sp(\widetilde{N}_0) \times U(\widetilde{N}_1)$. As in

[26] these transition was interpreted as a rearrangement of the compact one-cycles on the matrix model curve. Since the matrix model curve for $SO/Sp$ gauge theories have $\mathbf{Z}_2$ symmetry, this rearrangement should occur in the way that keeps this $\mathbf{Z}_2$ symmetry.

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